Further reducing this form, $s^3-3ls+m=0$, by the substitution $s=\nu W$, we have:

$$W^3-\frac{3l}{\nu^2}W+\frac{m}{\nu^3}=0.$$

Setting $l=\nu^2$, we have the form:

(8) $$W^3-3W\pm K=0, \quad \text{where} \quad K=m/l^{3/2}.$$

There are four cases. If $l=+\lambda$, $m=+\mu$; or $l=+\lambda$, $m=-\mu$, where λ and μ are both positive quantities, we have:

(8A) $$W^3-3W\pm\rho=0, \quad \text{where} \quad \rho=\mu/\lambda^{3/2}.$$

If however $l=-\lambda$, $m=+\mu$; or $l=-\lambda$, $m=-\mu$, where λ and μ are both positive quantities, we have

(8B) $$W^3-3W\pm\rho i=0, \quad \text{where} \quad \rho=\mu/\lambda^{3/2}.$$

Employing Vieta's transformation in the simplest form

(9) $$W=z+(1/z) \quad \text{where} \quad W=u+iv, \quad z=x+iy$$

we have

(10) $$u+iv=x+iy+(x+iy)^{-1}.$$

Solving we obtain:

(11) $$u=\frac{x(x^2+y^2+1)}{x^2+y^2}, \quad v=\frac{y(x^2+y^2-1)}{(x^2+y^2)}.$$

Clearly

$$W=\frac{x(x^2+y^2+1)}{x^2+y^2}+\frac{y(x^2+y^2-1)}{x^2+y^2}i$$

is invariant under the substitution

$$x=\frac{x_1}{(x_1^2+y_1^2)}, \quad y=\frac{-y_1}{(x_1^2+y_1^2)}$$

as is otherwise obvious, by the substitution $z_1=1/z$ in the formula $W=z+(1/z)$.

To get the inverse transformation, we employ $z+(1/z)=u+iv$, which gives

(11A) $$z\equiv x+iy=\tfrac{1}{2}\{u+iv\pm[(u^2-v^2-4)+2uvi]^{1/2}\}.$$

Setting

$$(\alpha+i\beta)^2=(u^2-v^2-4)+2uvi$$

we obtain

(12) $$\alpha^2=\tfrac{1}{2}\{(u^2-v^2-4)+[u^4+v^4+16+2u^2v^2-8u^2+8v^2]^{1/2}\},$$
$$\beta^2=\tfrac{1}{2}\{-(u^2-v^2-4)+[u^4+v^4+16+2u^2v^2-8u^2+8v^2]^{1/2}\}.$$

Hence from

$$x=\tfrac{1}{2}(u\pm\alpha), \qquad y=\tfrac{1}{2}(u\pm\beta),$$

with α and β from (12), we have

$$x=\frac{(\sqrt{2})u\pm\left\{[u^4+v^4+16+2u^2v^2-8u^2+8v^2]^{1/2}+(u^2+v^2-4)\right\}^{1/2}}{2\sqrt{2}},$$

(13)

$$y=\frac{(\sqrt{2})v\pm\left\{[u^4+v^4+16+2u^2v^2-8u^2+8v^2]^{1/2}-(u^2+v^2-4)\right]^{1/2}}{2\sqrt{2}}.$$

Making the substitution $W=u+iv$ in equation (8A), we have

$$u^3-3uv^2-3u\pm\rho=0, \quad v(3u^2-v^2-3)=0. \tag{14}$$

Plotting, in either case, the paired loci, we obtain in general three points of intersection; (u_1, v_1), (u_2, v_2), and (u_3, v_3). These supply the three roots W_1, W_2, W_3 of the original equation. By this method we are spared the necessity of plotting nonic curves. The three values of (u, v), set in equations (13), furnish the required values of x and y, which likewise furnish the three values of z.

For the other case, $W^3-3W\pm\rho i=0$, we have

$$u(u^2-3v^2-3)=0, \quad v^3-3u^2v+3v\pm\rho=0. \tag{15}$$

We then proceed as in the former case.

3. *Examples*:

A. Given $t^3+6t^2+9t+4=0$. Making the substitution $t=W-2$, we have $W^3-3W+2=0$. Using the substitution $W=u+iv$, we have

$$u^3-3uv^2-3u+2=0, \quad v(3u^2-v^2-3)=0.$$

The plotted or solved values for the intersections are (1, 0), (1, 0), (−2, 0). Hence the values of W are 1, 1, −2; and therefore the roots of the original equation are −1, −1, −4. The first curve above may be plotted by points from the equation,

$$v=\pm[(u^3-3u+2)/3u]^{1/2}.$$

The second curve consists of the straight line $v=0$ and the hyperbola, $u^2-\frac{1}{3}v^2=1$.

B. Given $t^3-6t^2+18t-40=0$. Making in order the substitutions $t=s+2$ and $s=+iW\sqrt{2}$, we have

$$s^3+6s-20=0 \quad \text{and} \quad W^3-3W-5i\sqrt{2}=0.$$

Equations (15) for this case are

$$u(u^2-3v^2-3)=0, \quad v^3+3(1-u^2)v+5\sqrt{2}=0.$$

Combining $u=0$ with the second equation, we have $v^3+3v+5\sqrt{2}=0$, giving one real root $v_1=-\sqrt{2}$. Hence $W_1=-i\sqrt{2}$. Combining $u^2-3v^2-3=0$ with the second equation, we have

$$8v^3+6v-5\sqrt{2}=0,$$

giving the single real value $\frac{1}{2}\sqrt{2}$. Since from the first equation $u = \pm 3\sqrt{2}/2$, the corresponding values of W are

$$W_2 = -\tfrac{1}{2}\sqrt{2}(3+i), \quad \text{and} \quad W_3 = -\tfrac{1}{2}\sqrt{2}(3-i).$$

The corresponding values of z, derived from equation (11A) are

$$\begin{aligned}
z_1 &= \tfrac{1}{2}\sqrt{2}(\sqrt{3}-1)i, \\
z_1' \equiv z_1^{-1} &= -\tfrac{1}{2}\sqrt{2}(\sqrt{3}+1)i, \\
z_2 &= \tfrac{1}{4}\sqrt{2}(\sqrt{3}+1)(\sqrt{3}+i), \\
z_2' \equiv z_2^{-1} &= \tfrac{1}{4}\sqrt{2}(\sqrt{3}-1)(\sqrt{3}-i); \\
z_3 &= -\tfrac{1}{4}\sqrt{2}(\sqrt{3}+1)(\sqrt{3}-i), \\
z_3' \equiv z_3^{-1} &= -\tfrac{1}{4}\sqrt{2}(\sqrt{3}-1)(\sqrt{3}+i).
\end{aligned}$$

From these six values of z, which are paired as indicated, the six pairs of values of x and y may be written down by inspection.

The geometrical aspects of the problem are clarified by figure 2. For instance, suppose we have z_1, z_2, and z_3, since only one set of three values is required. Choosing z_2, find its inverse point with reference to the unit circle, and then reflect this inverse point in the axis of reals. Complete the parallelogram formed by Oz_2 and Oz_2', giving the point W_2, as indicated by the formula

$$W_2 = z_2 + z_2^{-1} = z_2 + z_2'.$$

4. *The Biquadratic.* Employing Heilermann's method, we write the biquadratic equation,

$$t^4 + a_1t^3 + a_2t^2 + a_3t + a_4 = 0 \tag{16}$$

in the form

$$t^4 + a_1t^3 + \lambda t^2 + (a_2 - \lambda)t^2 + a_3t + a_4 = 0. \tag{16$'$}$$

Making the substitutions $x = t^2$, $y = t$, we have

$$F(x\ y) \equiv x^2 + a_1xy + a_2x + a_3y + a_4 + \lambda(y^2 - x) = 0. \tag{17}$$

We may now determine the parameter λ, so as to make the polynomial $F(x, y)$ break up into two linear factors. The required condition is

$$\begin{vmatrix} 2 & a_1 & a_2 - \lambda \\ a_1 & 2\lambda & a_3 \\ a_2 - \lambda & a_3 & 2a_4 \end{vmatrix} = 0$$

or

$$\lambda^3 - 2a_2\lambda^2 + (a_1a_3 + a_2^2 - 4a_4)\lambda + (a_1^2a_4 - a_1a_2a_3 + a_3^2) = 0.$$

The solution of this cubic in λ gives three values which, set in equation (17), give three quadratics, each of which breaks up into linear factors. Suppose, for example, that for one of the values of λ equation (17) takes the form

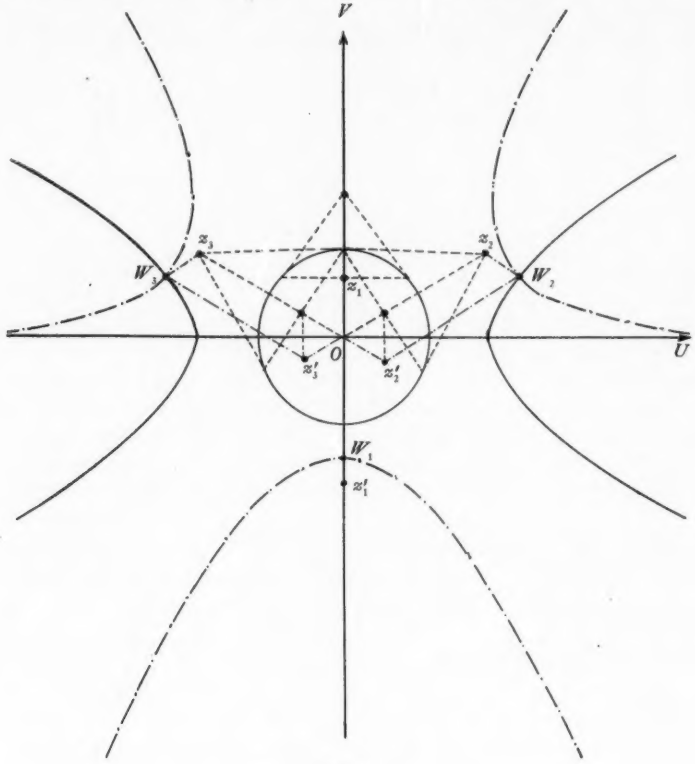

FIG. 2

$$(x + b_1y + c_1)(x + b_2y + c_2) = 0$$

or

$$(t^2 + b_1t + c_1)(t^2 + b_2t + c_2) = 0$$

or

$$\{(t - t_1)(t - t_2)\}\{(t - t_3)(t - t_4)\} = 0$$

For the three values of λ, the four roots t_1, t_2, t_3, t_4 are paired as follows:

$$t_1, t_2; t_3, t_4$$
$$t_1, t_3; t_2, t_4$$
$$t_1, t_4; t_2, t_3$$

5. *Geometrical Solution.* Consider the equation

$$(x^2 + a_1xy + a_2x + a_3y + a_4) + \lambda(y^2 - x) = 0.$$

This is a single parameter family of conics, passing through the points of intersection of the two conics

$$x^2 + a_1xy + a_2x + a_3y + a_4 = 0 \tag{18}$$

and

$$y^2 - x = 0. \tag{19}$$

A geometric solution without employing the cubic in λ is afforded directly, the four solutions of the original biquadratic in t being the ordinates of the four points of intersection of the plotted conics (18) and (19), the latter being a base parabola.

A second method is to find the intersections of a single pair of lines

$$x + b_1y + c_1 = 0, \quad x + b_2y + c_2 = 0,$$

found by the use of one solution of the cubic in λ, with the base parabola (19).

A third method is to find the intersections of two pairs of such lines, derived from two different solutions of the cubic in λ.

If the conics (18) and (19) intersect in four points, the three pairs of lines derived from the solutions of the cubic in λ form a complete quadrangle on the four points.[2]

Example. Given $t^4 - 2t^3 - t^2 + 2t = 0$. From the equation $\lambda^3 + 2\lambda^2 - 3\lambda = 0$ we have $\lambda = 0, -1, 3$. For $\lambda = 0$,

$$x^2 - 2xy - x + 2y = 0 \quad \text{or} \quad (x - 1)(x - 2y) = 0.$$

Making this simultaneous with $y^2 - x = 0$, we find for the values of y, or t, $0, +1, -1, 2$.

TWO NOTES ON INSTRUCTION IN MECHANICS

By OLIVER D. KELLOGG, Harvard University

1. *On the "principle" that Newton's second law, together with the initial position and velocity, determine the motion of a particle.*

The following question appeared in a recent examination in mechanics: *Prove that the motion of a particle under a central force takes place in a fixed plane.* A number of answers were substantially as follows. "Consider the plane containing the given center, the initial position, and the initial velocity of the

[2] Consult Archibald Henderson, *Observations on simultaneous quadratic equations*, in this Monthly, vol. 35 (1928), pp. 337–346. A full discussion of the various cases arising from degeneration of the complete quadrangle is found in that paper.

particle. As the force lies in this plane, there is no normal acceleration, so that the particle remains in the plane."

The question arose as to whether this reasoning is valid. We are not here concerned with the failure of the students to notice that if the velocity is initially toward the center, the plane in question is not determined. Let us suppose the statement of the question precluded such an initial velocity. Since the theorem is true, it may appear a little difficult to assure oneself as to the propriety of the reasoning. This may, however, be done as follows. If we can construct a field of force (not, of course, central), to which the reasoning applies, and yet in which the motion is not necessarily plane, we shall know that the reasoning is not valid.

Now the solutions of the differential equations of motion are known to be unique in case the components of force on the coördinate axes satisfy a Lipschitz condition. And if the plane of the initial velocity be taken as the (x, y)-plane, and there is no component of force normal to that plane as long as the particle remains in the plane, then $z=0$ is part of *one* possible solution. We must therefore seek a field of force in which a Lipschitz condition is *not* satisfied. Such a one is given by the components

$$X = -x, \quad Y = -y, \quad Z = 12\,|z|^{1/2}.$$

If we take the mass of the particle as 1, and suppose that for $t=0$ it is at $(1, 0, 0)$, with the velocity $(0, 1, 0)$, the equations of motion and initial conditions will be found to be satisfied by

$$x = \cos t, \quad y = \sin t, \quad z = t^4,$$

and these equations do not represent a plane curve, in spite of the fact that there is no component of force normal to the plane $z=0$ which contains the particle and its velocity for $t=0$. Thus the students' "proof" is not a proof.

It is not contended that no credit should be given the indicated replies to the question. A discussion of the subject might be interesting, but this was not my intention in writing this paper.

As to the "principle" cited above, the example shows that it is a dangerous one. It may be urged that nature does not construct fields of force like the one given. But all our mathematical treatment of nature is based on idealizations, and there is no guarantee that an idealization might not lead to a field of force which does not have the Lipschitz property. Certainly, the "principle" as stated is not mathematically sound.

II. *On the Principle of Virtual Work.*

There are few topics in mechanics on which there is more lax writing and thinking than the principle of virtual work. We propose here, not so much to go into all the details of a precise formulation, as to point out an apparent failure of the principle.

Coördinates of a mechanical system must be defined. A *virtual displacement* of the system is then defined by a *variation* of the coördinates. A variation of the

coördinates is a system of *differentials*, satisfying the linear equations obtained by differentiating the equations of constraint (if the constraints are holonomic —otherwise satisfying the non-holonomic equations of constraint). Whether a virtual displacement is "actual," "real," or "possible," is utterly beside the point. If an equation of constraint is non-linear in the coördinates, and is not identically satisfied by them, then, in general, a virtual displacement does not correspond to a possible configuration of the system.

The constrains are assumed to be *smooth*, which means that the forces of constraint do no work for any virtual displacement. Calling all other forces applied, we may state the principle as follows: *A necessary and sufficient condition that the system be in equilibrium is that the virtual work of the applied forces vanishes for every virtual displacement.*

Some years ago, Professor W. F. Osgood called my attention to the following example. Imagine a weightless circular hoop, in a vertical plane, constrained to roll on the under side of an incline plane, without slipping. To a point of the hoop is attached a heavy particle. Study the possibility of equilibrium.

If we take the x-axis in the plane, and downward, and the y-axis perpendicular and also downward, it is natural to take as coördinate the angle through which the radius of the hoop through the particle has rolled from its position when the particle is in contact with the plane. The coördinates of the particle are then given by the usual equations of the cycloid:

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta),$$

and we find for the virtual work corresponding to a variation $\delta\theta$

$$\delta W = 2wa \sin\tfrac{1}{2}\theta \cos(\tfrac{1}{2}\theta - \alpha)\delta\theta,$$

where α is the inclination of the plane, and w the weight of the particle. The principle of virtual work tells us that a necessary and sufficient condition for equilibrium is the vanishing of the sine or the cosine in this expression. The vanishing of the cosine gives correct results: the particle is in equilibrium at every point of the path of minimum height. The vanishing of the sine, however, gives us the points of contact of the particle with the plane, which I think most of us would regard as spurious positions of equilibrium.

Perhaps an even more striking example is found in a bead, free to slide under gravity on a vertical wire. If z is the distance of the bead above an arbitrary point of the wire, and we take as the coördinate (and surely generalized coördinates are admissible) of the particle, q, defined by $z = q^3$, then the virtual work of the weight w of the bead is

$$\delta W = -3wq^2\delta q,$$

and this vanishes at $q = 0$ for every virtual displacement. We should thus be led to infer that *any given position of the bead is one of equilibrium!*

The second example will perhaps raise a suspicion as to where the difficulty lies, because of the artificiality of the coördinate used. Indeed this is just the point. It is not the principle that has failed us, it is the coördinates employed —in spite of the fact that in the first example θ is the most natural one to select. However, if we had used y instead, we should not have obtained the spurious positions of equilibrium.

The obvious way out is to lay proper restrictions on the choice of systems of coördinates. A sufficient restriction is the following. *A system of coördinates is not admissible unless to every variation of the coördinates* (other than that in which all differentials vanish) *there corresponds a change of position of at least one point of the system.* This may be formulated analytically by selecting a set of points of the system (assumed to have a finite number of degrees of freedom) such that the positions of these points fix those of all points of the system. Then the matrix of the partial derivatives of the cartesian coördinates of these points with respect to the chosen coördinates of the system must be of maximum rank.

Another way of avoiding the trouble lies in a change of definition of virtual displacement. We define *virtual velocities* as any system of velocities of the points of the system compatible with the constraints. A virtual displacement is then any system of displacements of the points of the system proportional to a system of virtual velocities. This procedure seems simpler. However, sooner or later the difficulties which coördinates may introduce must be faced.

THE HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

By MEYER SALKOVER, University of Cincinnati

This note deals with the solution of the differential equation in one dependent and one or two independent variables, linear, with constant coefficients, in the dependent variable and its derivatives, there being no term in the equation free from the latter quantities. By "homogeneous" is here meant the last requirement. The treatment is elementary, inductive, and characterized by the avoidance of artificial limiting processes, abstruse symbols (except in the last paragraph), and the customary auxiliary theorems. Moreover, the parallelism between the ordinary and the partial differential equation of the type studied will be clearly exhibited.

1. *The Ordinary Differential Equation*

The ordinary differential equation

$$(D - a)^m z = 0, \tag{1}$$

where D means the operator d/dx, the exponent m is a positive integer, any

power of D meaning the derivative of the corresponding order, and where a is a constant, may be solved as follows: We put

$$(D - a)z = R. \tag{2}$$

To find a serviceable expression for R, we might, in the usual way, apply the integrating factor e^{-ax}; or, equivalently, write (2) as

$$(d/dx)(\log z) - a = R/z$$

or

$$(d/dx)[\log (ze^{-ax})] = R/z;$$

whence

$$(D - a)z = R = z(d/dx)[\log (ze^{-ax})] = e^{ax}(d/dx)(ze^{-ax}). \tag{3}$$

By induction, we obtain

$$(D - a)^m z = e^{ax}(d^m/dx^m)(ze^{-ax}). \tag{4}$$

Hence (1) reduces to

$$(d^m/dx^m)(ze^{-ax}) = 0,$$

which, after repeated integration, yields

$$z = e^{ax}(C_0 + C_1x + \cdots + C_{m-1}x^{m-1}), \tag{5}$$

where the C's are arbitrary constants.

Alternative forms for (3) and (4) are

$$(D - a)z = [(d/dx)(e^{C-ax}z)]_{C=ax} \tag{3'}$$

and

$$(D - a)^m z = [(d^m/dx^m)(e^{C-ax}z)]_{C=ax}. \tag{4'}$$

These identities will be cited later when (1) will be treated as a conditioned form of the partial differential equation of the corresponding type.

The general ordinary homogeneous linear differential equation with constant coefficients, which, by the theory of equations and the properties of the derivative, may be written in operationally factored form

$$(D - a_1)^{m_1}(D - a_2)^{m_2} \cdots (D - a_n)^{m_n}z = 0, \tag{6}$$

is easily solved by iteration of the above process. Let us put

$$v_1 = (D - a_2)^{m_2}(D - a_3)^{m_3} \cdots (D - a_n)^{m_n}z.$$

From (5),

$$v_1 = (C_{10} + C_{11}x + \cdots + C_{1m_1-1}x^{m_1-1})e^{a_1x}. \tag{7}$$

We then introduce

$$v_2 = (D - a_3)^{m_3} \cdots (D - a_n)^{m_n}z,$$

so that

$$v_1 = (D - a_2)^{m_2} v_2;$$

we substitute in (7), use (4), divide by $e^{a_1 x}$ and integrate. Repeating this process, we ultimately find the solution of (6) to be

$$z = \sum_{i=1}^{n} (C_{i0} + C_{i1}x + \cdots + C_{im_i-1}x^{m_i-1})e^{a_i x}. \tag{8}$$

The regular practice of obtaining (8) by adding the particular solutions arising from the separate factors of (6) is more elegant, though not essentially simpler. It requires, moreover, a digression to establish an auxiliary theorem.

The same procedure is applicable to the non-homogeneous equation, but here symbolic manipulation in some cases saves considerable time.

2. *The Partial Differential Equation*

In all elementary texts[1] to which the writer has referred, linear partial differential equations with constant coefficients are solved either symbolically by exponential operators suggested by the theory of the ordinary equation, or stepwise, without the aid of induction, or, in more complicated cases, by combining series of exponential functions with coefficients that are arbitrary constants into arbitrary functions. Here the subject is developed inductively along the lines of our treatment of the ordinary equation but without artificially and symbolically using previous results.

A. Binomial Factors of the Form $D_x - aD_y$. Let us consider first the equation

$$(D_x - aD_y)^m z(x, y) = 0, \tag{9}$$

where D_x and D_y are the operators $\partial/\partial x$ and $\partial/\partial y$, respectively. We put

$$(D_x - aD_y)z = R(x, y).$$

By the method of Lagrange,

$$\frac{dx}{1} = \frac{dy}{-a} = \frac{dz}{R};$$

whence

$$ax + y = C$$

and

$$z(x, C - ax) = \int^x R(\xi, C - a\xi)d\xi + f(C),$$

f being an arbitrary function of its argument. Thus we find

$$R(x, y) = (D_x - aD_y)z(x, y) = [\partial z(x, C - ax)/\partial x]_{C=ax+y}. \tag{10}$$

[1] e. g., Forsyth, *Differential Equations*, 4th edition, pp. 512, 516, 517; Cohen, *Differential Equations*, pp. 239–242, 246–247.

This is the analogue of (3) or (3′); in fact, (3′) may be derived from (10) by specialization as follows:

If $(D_x - aD_y)z$ reduces to $(D_x - a)z$,

$$D_y z = z;$$

and on integration

$$z = e^y f(x).$$

Substituting this in (10) gives us

$$R = [\partial(f(x)e^{C-ax})/\partial x]_{C=ax+y}. \tag{11}$$

The transition to the standpoint of the ordinary differential equation is easily made by considering y as an arbitrary constant. If we write K for the difference between y and C, the right side of (11) becomes

$$[d(ze^{K-ax})/dx]_{K=ax},$$

as in (3′).

By induction from (10) we find

$$(D_x - aD_y)^m z(x, y) = [\partial^m z(x, C - ax)/\partial x^m]_{C=ax+y}. \tag{12}$$

This is the analogue of (4) or (4′), which may be derived from (12) by specialization.

By (12), (9) becomes

$$[\partial^m z(x, C - ax)/\partial x^m]_{C=ax+y} = 0. \tag{13}$$

If in (9) the independent variable y had been replaced by $C-ax$, we would have obtained instead of (13),

$$\partial^m z(x, C - ax)/\partial x^m = 0,$$

whence, by repeated integration,

$$z(x, C - ax) = f_0(C) + xf_1(C) + \cdots + x^{m-1}f_{m-1}(C),$$

where the f's are arbitrary functions. Restoring $y = C - ax$, we obtain

$$z(x, y) = f_0(ax + y) + xf_1(ax + y) + \cdots + x^{m-1}f_{m-1}(ax + y) \tag{14}$$

as the solution of (9).

By the same sort of specialization as was applied to (10), (14) can be converted into (5), the solution of the corresponding ordinary differential equation (1).

The homogeneous linear partial differential equation with constant coefficients in which the derivatives are all of the same order and in which the derivative of highest order with respect to x is of zero order with respect to y may be written in the following general form:

$$(D_x - a_1D_y)^{m_1}(D_x - a_2D_y)^{m_2} \cdots (D_x - a_nD_y)^{m_n} z = 0. \tag{15}$$

It may be solved (see the indicated solution of (6)) by repeated use of (12), the operational binomial powers being unraveled one at a time, starting with $(D_x - aD_y)^{m_1}$. We thus find for the solution of (15) the sum of n expressions of the form (14), in the successive arguments

$$a_1x + y,\ a_2x + y,\ \cdots,\ a_nx + y.$$

B. Other Linear Factors. If operational linear factors such as are excluded by the restrictions of the previous section exist, they, too, can be brought within the scope of our method. Suppose that the repeated general linear factor $(D_x - aD_y - b)^m$ occurs. From the foregoing we may infer that its effect on the solution may be ascertained by solving the equation

$$(D_x - aD_y - b)^m z = 0. \tag{16}$$

Putting

$$(D_x - aD_y - b)z = R(x, y)$$

we find, from (10),

$$R(x, y) = [\partial z(x, C - ax)/\partial x]_{C=ax+y} - bz(x, y)$$

or

$$\begin{aligned} R(x, C - ax) &= \partial z(x, C - ax)/\partial x - bz(x, C - ax) \\ &= e^{bx}\partial[z(x, C - ax)e^{-bx}]/\partial x, \end{aligned}$$

(using (3)). By induction,

$$(D_x - aD_y - b)^m z = e^{bx}[\partial^m\{z(x, C - ax)e^{-bx}\}/\partial x^m]_{C=ax+y}.$$

Whence, returning to (16) and proceeding as with (13), we obtain the solution:

$$z(x, y) = e^{bx}[f_0(ax + y) + xf_1(ax + y) + \cdots + x^{m-1}f_{m-1}(ax + y)]. \tag{17}$$

It will be noticed that the arbitrary functions arise naturally instead of, as in the usual treatment, being built up from series of exponentials having as coefficients arbitrary constants.

For every factor of the type (16) in a homogeneous linear partial differential equation with constant coefficients, there will evidently be a term of the type (17) in the solution.

If repeated linear factors free from D_x, such as $(D_y - b)^m$, occur, the corresponding terms in the solution are readily seen from (16) and (17), interchanging x and y and putting $a = 0$, to have the form

$$e^{by}[f_0(x) + yf_1(x) + \cdots + y^{m-1}f_{m-1}(x)].$$

C. A Symbolic Solution. Our results so far obtained may be extended symbolically to more difficult cases. Take, for instance, the differential equation for the linear flow of heat (without imposing conditions on the function sought),

$$(D_x^2 - b^2D_y)z = 0,$$

treated by Forsyth[2] by exponential operators. This may be written

$$(D_x - bD_y^{-1/2}D_y)(D_x + bD_y^{-1/2}D_y)z = 0.$$

Thus written, it formally comes under (15), with $a_1 = bD_y^{-\frac{1}{2}}$ and $a_2 = -bD_y^{-\frac{1}{2}}$. The symbolic solution is then

$$z = f_1(bD_y^{-1/2}x + y) + f_2(-bD_y^{-1/2}x + y).$$

If the arbitrary functions are expanded into Taylor's series in powers of the first term of each argument, the powers of $D_y^{-\frac{1}{2}}$ operating on the derivatives of $f_i(y)(i=1, 2)$, and then, as in Forsyth, the arbitrary functions are renamed to efface the obscure operator $D_y^{-1/2}$, the series for z in two arbitrary functions is obtained. The equivalent series in one arbitrary function is found by writing the equation in the form

$$(D_y - b^{-2}D_x \cdot D_x)z = 0,$$

which yields, symbolically,

$$z = f(b^{-2}D_x y + x);$$

and this is to be expanded into a Taylor's series in powers of the first term of the argument, the powers of D_x operating on the derivatives of $f(x)$.

SCALAR EXTENSIONS OF AN ORTHOGONAL ENNUPLE OF VECTORS

By A. D. MICHAL, California Institute of Technology

A general theory of scalar extensions of tensors has been developed by the author in another paper.[1] The aim of the present note is to single out an important special case and to give the simpler details independently of the general theory.

Let

$$g_{\alpha\beta}dx^\alpha dx^\beta; \quad g_{\alpha\beta} = g_{\beta\alpha} \tag{1}$$

be the fundamental metric differential form of an n-dimensional Riemann space. For the sake of simplicity we shall assume that the form (1) is positive definite.[2] Throughout our paper we shall adopt the convention of letting the repetition of an index in a term denote summation with respect to that index over all integral values $1, 2, \cdots, n$. Furthermore by x^i we shall understand the ith coordinate of a point $(x^1, x^2, \cdots x^i, \cdots, x^n)$ and not the ith power of a variable x.

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[2] Loc. cit., pp. 522–524.

[1] To be published in The Tôhoku Mathematical Journal.

[2] Obvious modifications will make our work valid for Eisenhart's more general metrics. See Eisenhart, *Riemannian Geometry* (1926), Chapt. III.

Let ${}_{(\alpha)}\xi^i(x^1, x^2, \cdots, x^n)$ denote the ith component of the αth contravariant vector in an orthogonal ennuple of contravariant vectors ${}_{(1)}\xi^i, {}_{(2)}\xi^i, \cdots, {}_{(n)}\xi^i$. The analytical expressions of these hypotheses are as follows:

(I) under an arbitrary analytic change of coordinates,

$$G : x^i = f^i(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n), \quad \left|\frac{\partial x^i}{\partial \bar{x}^j}\right| \neq 0, \tag{2}$$

the components of the αth vector ${}_{(\alpha)}\xi^i$ undergo a transformation[3]

$${}_{(\alpha)}\xi^i(x^1, \cdots, x^n) = {}_{(\alpha)}\bar{\xi}^\sigma(\bar{x}^1, \cdots, \bar{x}^n)\frac{\partial x^i}{\partial \bar{x}^\sigma}. \tag{3}$$

(II) the vectors ${}_{(\alpha)}\xi^i$ satisfy the invariant conditions

$$g_{\sigma\tau\,(\alpha)}\xi^\sigma{}_{(\beta)}\xi^\tau = \delta_{\alpha\beta}, \tag{4}$$

where

$$\delta_{\alpha\beta}\begin{cases} = 1 \text{ if } \alpha = \beta \\ = 0 \text{ if } \alpha \neq \beta. \end{cases}$$

Define ${}_{(\alpha)}\xi_i$ by

$${}_{(\alpha)}\xi_i = g_{ij\,(\alpha)}\xi^j. \tag{5}$$

Then it easily follows from (4) that ${}_{(\alpha)}\xi_i$ is the cofactor of ${}_{(i)}\xi^\alpha$ in the determinant of ${}_{(i)}\xi^\alpha$, divided by the determinant of ${}_{(i)}\xi^\alpha$.

Let $\Gamma^i_{\alpha\beta}$ stand for the Christoffel symbols of the second kind

$$\Gamma^i_{\alpha\beta} = \frac{1}{2}g^{i\sigma}\left(\frac{\partial g_{\sigma\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right). \tag{6}$$

As is well known the law of transformation of the $\Gamma^i_{\alpha\beta}$ under an arbitrary transformation G is given by

$$\bar{\Gamma}^\sigma_{\alpha\beta}(\bar{x}^1, \cdots, \bar{x}^n)\frac{\partial x^i}{\partial \bar{x}^\sigma} = \Gamma^i_{\lambda\mu}\frac{\partial x^\lambda}{\partial \bar{x}^\alpha}\frac{\partial x^\mu}{\partial \bar{x}^\beta} + \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta}. \tag{7}$$

Let x^i be an arbitrarily given coordinate system and the point P an arbitrarily given point in our space with coordinates $x^i = q^i$. Define a coordinate system S_1, with coordinates y^i, implicitly by

$$x^i = q^i + \left[{}_{(\alpha)}\xi^i(x^1, \cdots, x^n)\right]_p y^\alpha - \frac{1}{2!}\left[\Gamma^i_{\lambda\mu\,(\alpha)}\xi^\lambda{}_{(\beta)}\xi^\mu\right]_p y^\alpha y^\beta. \tag{8}$$

Since the determinant of ${}_{(\alpha)}\xi^i$ does not vanish at the point P, it follows that (8) is a reversible transformation in the neighborhood of the point P. Let a star

[3] We shall always understand that any free index (not summed out) can take on any integral value $1, 2, \cdots, n$.

over a component denote the evaluation of that component in a coordinate system S_1. If we now consider the particular transformation G, given by (8), in the transformation laws (3) and (7), we obtain after some easy reductions the results

$$(9) \quad (a) \quad \left[{}^{*}_{(\alpha)}\xi^{i}(y^1, \cdots, y^n)\right]_0 = \delta^{i}_{\alpha}; \quad (b) \quad \left[{}^{*}\Gamma^{i}_{\alpha\beta}(y^1, \cdots, y^n)\right]_0 = 0,$$

where δ^{i}_{α} is unity or zero according as $i=\alpha$ or $i\neq\alpha$ and where the suffix 0 denotes evaluation for $y^i=0$. Now a coordinate system y^i for which (9) (b) is satisfied is known as a geodesic coordinate system. Hence S_1 is a special geodesic coordinate system.

Consider an arbitrary transformation G on the coordinates x^i. Then as before, the new coordinate system $\bar{x}^i$ and the *same* point P with new coordinates $\bar{x}^i=\bar{q}^i$ can define implicitly a coordinate system $\bar{y}^i$ with corresponding properties (9) by means of the transformation

$$(10) \qquad \bar{x}^i = \bar{q}^i + \left[{}_{(\alpha)}\bar{\xi}^{i}(\bar{x}^1, \cdots, \bar{x}^n)\right]_p \bar{y}^{\alpha} - \frac{1}{2!}\left[\bar{\Gamma}^{i}_{\lambda\mu\,(\alpha)}\bar{\xi}^{\lambda}{}_{(\beta)}\bar{\xi}^{\mu}\right]_p \bar{y}^{\alpha}\bar{y}^{\beta}.$$

If we write (3) and (7) as transformation laws between components in the y^i coordinate system and the corresponding components in the $\bar{y}^i$ coordinate system we obtain by means of (9) and a similar set of formulas for the $\bar{y}^i$ coordinate system the results

$$(11) \qquad \left(\frac{\partial y^i}{\partial \bar{y}^{\alpha}}\right)_0 = \delta^{i}_{\alpha}; \quad \left(\frac{\partial^2 y^i}{\partial \bar{y}^{\alpha}\partial \bar{y}^{\beta}}\right)_0 = 0.$$

We see from (5) that ${}_{(\alpha)}\xi_i$ is the ith component of the αth covariant vector of the ennuple of covariant vectors ${}_{(1)}\xi_i, \cdots, {}_{(n)}\xi_i$. Let y^i and $\bar{y}^i$ be geodesic coordinates of type S_1 with the same origin P. On differentiating the set of equations

$$(12) \qquad {}^{*}_{(\alpha)}\bar{\xi}_i(\bar{y}^1, \cdots, \bar{y}^n) = {}^{*}_{(\alpha)}\xi_{\sigma}(y^1, \cdots, y^n)\frac{\partial y^{\sigma}}{\partial \bar{y}^i}$$

and making use of (11) we obtain at the common origin of the coordinate systems y^i and $\bar{y}^i$

$$(13) \qquad \left(\frac{\partial\, {}^{*}_{(\alpha)}\bar{\xi}_i}{\partial \bar{y}^j}\right)_0 = \left(\frac{\partial\, {}^{*}_{(\alpha)}\xi_i}{\partial y^j}\right)_0.$$

Hence the functions $\gamma_{\alpha ij}(x^1, \cdots, x^n)$ defined by

$$(14) \qquad \left[\gamma_{\alpha ij}(x^1, \cdots, x^n)\right]_p = \left(\frac{\partial\, {}^{*}_{(\alpha)}\xi_i}{\partial y^j}\right)_0$$

are absolute scalar functions. This set of scalars will be called the *first scalar extension of the vectors* ${}_{(\alpha)}\xi_i$.

THEOREM I: *The scalars γ_{ijk} in the first scalar extension of the covariant vectors ${}_{(i)}\xi_j$ are identical with the Ricci coefficients of rotation of the orthogonal ennuple.*

To prove this interesting theorem, we write down the equations

$$(15) \qquad {}^{*}_{(i)}\xi_j(y^1, \cdots, y^n) = {}_{(i)}\xi_\lambda(x^1, \cdots, x^n)\frac{\partial x^\lambda}{\partial y^j},$$

where y^i is a geodesic coordinate system of type S_1, differentiate with respect to y^k and then evaluate the results at the origin of the coordinate system y^i. Finally by substituting for

$$\left(\frac{\partial x^i}{\partial y^j}\right)_0 \text{ and } \left(\frac{\partial^2 x^i}{\partial y^j \partial y^k}\right)_0$$

their expressions as given by (8), we obtain

$$(16) \qquad \gamma_{ijk} = \left(\frac{\partial\, {}_{(i)}\xi_\lambda}{\partial x^\mu} - {}_{(i)}\xi_\sigma \Gamma^\sigma_{\lambda\mu}\right) {}_{(j)}\xi^\lambda\, {}_{(k)}\xi^\mu.$$

Up to this point we have made use only of the existence and continuity of the first partial derivatives of the components of the vectors in the ennuple of vectors. Assume now the existence and continuity of all the second partial derivatives of the components of the vectors in the given orthogonal ennuple. On adding

$$(17) \qquad -\frac{1}{3!}\left[\Gamma_{\lambda\mu\nu\,(\alpha)}\xi^\lambda\, {}_{(\beta)}\xi^\mu\, {}_{(\gamma)}\xi^\nu\right]_p y^\alpha y^\beta y^\gamma$$

to the corresponding right hand sides of (8), the modified transformation of coordinates (8) defines implicitly another special type, say S_2, of geodesic coordinates y^i. In (17), the set of functions $\Gamma^i{}_{\lambda\mu\nu}$ are defined by

$$\Gamma^i_{\lambda\mu\nu} = \frac{1}{3}P\left(\frac{\partial \Gamma^i_{\lambda\mu}}{\partial x^\nu} - \Gamma_{\sigma\mu}\Gamma^\sigma_{\lambda\nu} - \Gamma^i_{\lambda\sigma}\Gamma^\sigma{}_\nu\right),$$

where P denotes the sum of the terms obtainable from the ones inside the parenthesis by permuting λ, μ, ν cyclically.[4]

The set of functions $g_{\alpha\beta\gamma\delta}(x^1, \cdots, x^n)$ defined by

$$(g_{\alpha\beta\gamma\delta})_p = \left(\frac{\partial^2 {}^{*}g_{\alpha\beta}}{\partial y^\gamma \partial y^\delta}\right)_0$$

with the aid of a geodesic coordinate system S_2 are absolute scalar functions. This set of scalars will be said to constitute the *second scalar extension of* $g_{\alpha\beta}$.

[4] Their law of transformation is given by O. Veblen and T. Y. Thomas in the Transactions of American Mathematical Society, vol. 25 (1923), p. 577.

By an obvious extension of the method of proof of Theorem I, one can demonstrate the validity of the following theorem.

THEOREM II: *If g_{ijkl} are the scalar functions of the second scalar extension of the metric tensor g_{ij}, then the set of functions $g_{ikjl}-g_{iljk}$ is identical with the set of the Ricci four-index scalars,*[5] $\gamma_{ij,kl}$.

A NOTE ON THE NUMERICAL COMBINATION OF PROBABILITY SERIES

By GEORGE C. CAMPBELL, University of Iowa

The object of this note is to give a practical method for the numerical combination of probability series.

Consider the two series, P_r' and P_r'', derived from the American Men mortality table, giving the probability of exactly r deaths from two homogeneous groups of 100 lives, age 20 and 30, respectively.

r	P_r'	P_r''	P_r'''
0	0.675 184	0.639 545	0.431 811
1	0.265 714	0.286 515	0.363 386
2	0.051 762	0.063 537	0.152 134
3	0.006 654	0.009 298	0.042 247
4	0.000 635	0.001 010	0.008 754
5	0.000 048	0.000 087	0.001 444
6	0.000 003	0.000 006	0.000 197
7	0.000 000	0.000 000	0.000 023
8			0.000 002
9			0.000 000

We ask for the series P_r''', giving the probability of exactly r deaths from the heterogeneous group of 200 lives composed of the two sub-groups. It is fairly obvious that

$$P_0''' = P_0' P_0'',$$
$$P_1''' = P_0' P_1'' + P_1' P_0'',$$
$$P_2''' = P_0' P_2'' + P_2' P_0'' + P_1' P_1'',$$
$$P_3' ' = P_0' P_3'' + P_3' P_0'' + P_2' P_1'' + P_1' P_2'',$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

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[5] For the definition of $\gamma_{ij,kl}$ see T. Levi-Civita, *The Absolute Differential Calculus* (1927), p. 278. For more general metrics, see L. P. Eisenhart, *loc. cit.*

where P_r''' is the sum of the products of all pairs of probabilities whose subscripts add up to the sum r. It is clear that this direct approach leads to an enormous amount of tabulation, even for series of moderate length.

Let us tabulate the series P_r' on a strip of paper and the series P_r'' in reverse order on another strip. Fasten the two strips with paper clips to a strip of card-board so that they will slide, side by side. Set P_0' opposite P_0''. Their product gives P_0'''. Then we slide the strips so that two pairs of lines are opposite each other, P_1' opposite P_0'' and P_0' opposite P_1''. If we do both of these multiplications on the computing machine without clearing the first product, we have the sum of the products on the machine, giving P_1'''. We continue to slide the strips by one line each time, and at each setting we multiply together all pairs of numbers opposite each other. By leaving the products on the machine, we obtain directly the sum of products which is P_r'''.

		. . . . .	
		. . . . .	
		0.063	537
		0.286	515
0.675	184	0.639	545
0.265	714		
0.051	762		
. . . . .			
. . . . .			

We note that this simple device carries us directly from the component series to the resultant series without tabulating any intermediate steps.

QUESTIONS AND DISCUSSIONS

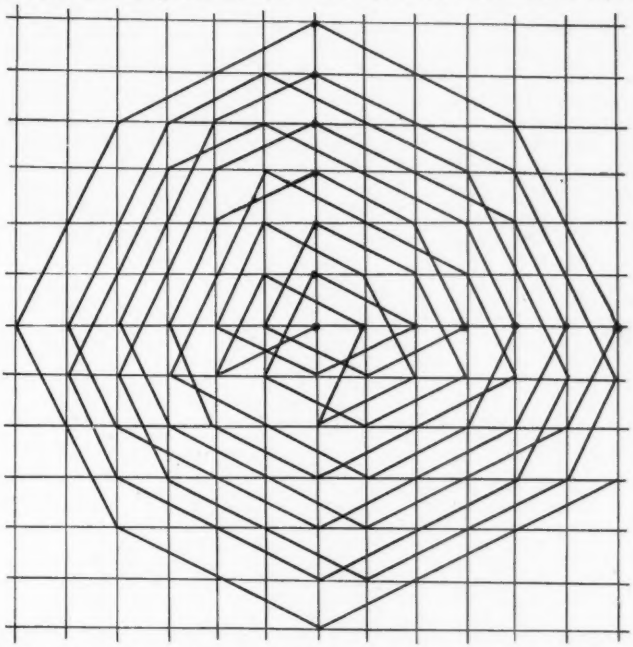
EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A RECREATION

By NORMAN ANNING, University of Michigan

The points of the plane which have integral coordinates form a countable set. Among the ways of counting them is that in which we proceed from each to the next by a knight's move. After a few irregular moves at the beginning the process can be systematized. The recreation consists in trying to find



a path which reduces to a minimum the number of irregular moves. Just where regularity sets in is open to question but please agree that the accompanying figure has 4 irregular moves. Similarly figures can be drawn for the half-plane and quarter-plane with irregularities as small as 8 and 13 respectively. It is not claimed that these numbers are the smallest of their classes; there's the puzzle.

A Note by the Editor

If one makes use of some of the well known configurations for the knight's tour in a square,[1] it is not difficult to show that the integral points in the whole

[1] See, for example, Dudeney's *Amusements in Mathematics*, pp. 102–103, 227–229.

plane can be counted in a sequence of knight's moves, and by an entirely systematic process involving no irregular moves whatever. (See figures 1, 2.) Indeed it is possible to do this in an infinite number of ways. The same statements

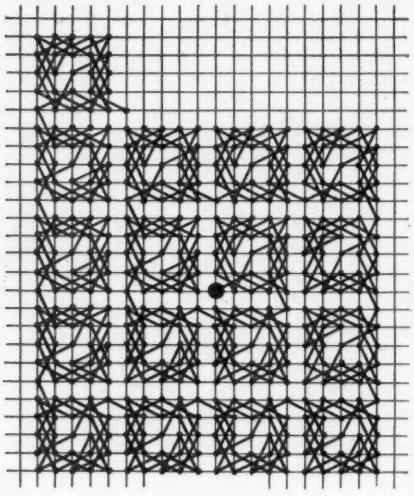


FIG. 1.

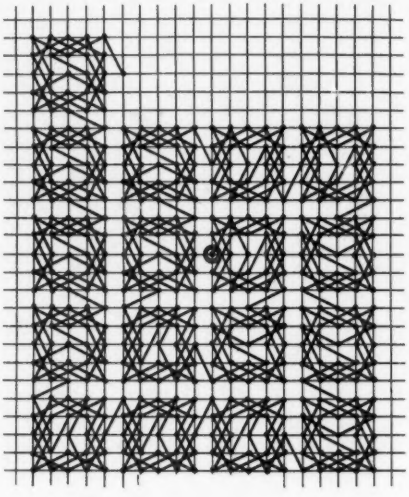


FIG. 2.

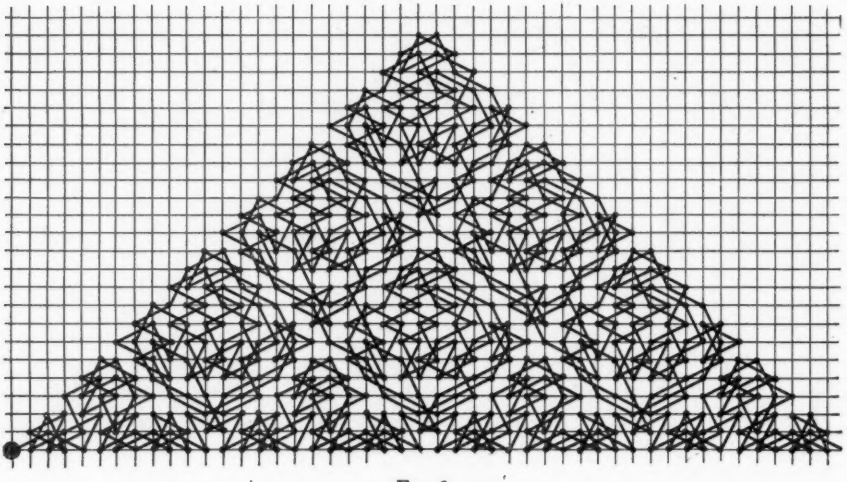


FIG. 3.

hold for half and quarter planes whose boundaries are parallel to the coordinate axes. It will be observed that the problem is very much changed by giving the boundaries of the half or quarter plane an orientation oblique to the coordinate

axes. This suggests the problem as to the possibility of counting the integer points in a sector of the plane. If the vertex of the sector is at the origin and the initial side coincides with the positive x-axis, the enumeration is impossible

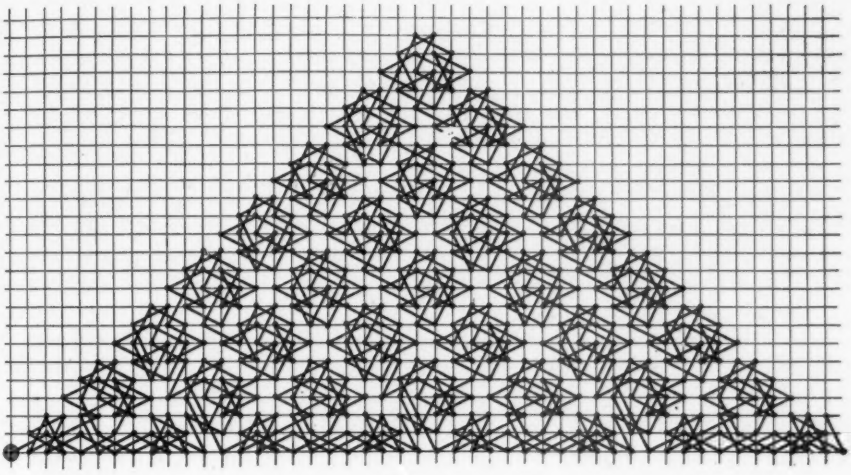


FIG. 4.

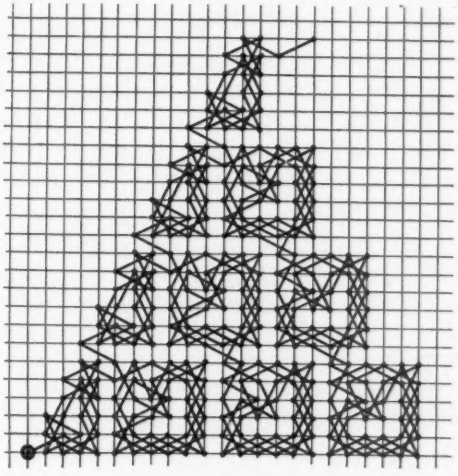


FIG. 5.

by any process systematic or otherwise, if the angle of the sector is less than 45°. For the 45° sector itself an infinite number of entirely systematic enumerations are possible (see figures 3, 4), and the same statement holds for the sector whose angle is $\tan^{-1} 2$ (see figure 5). More precisely, if as in figures 3 and 4 two

different designs are discovered each of which by suitable iteration effects the required enumeration in some sector, and if these designs, though different, are such that the areas occupied by the fundamental pattern and of each successive iteration in the one design are congruent respectively to the areas of the fundamental pattern and each successive iteration in the other, then arbitrary alternations from one pattern to the other are possible, and it can accordingly be asserted that there exists a non-denumerably infinite number of ways of counting the points in that sector, of which at least a denumerably infinite number are systematic. For sectors of 360°, 180°, 90°, 45°, $\tan^{-1} 2$, $\tan^{-1} 3/2$, and $\tan^{-1} 4/3$, such is actually the case.

It is probable that for any sector whose angle is not less than 45°, the points with integral coordinates, inside and on the boundary, can be enumerated in a sequence of knight's moves, and in a non-denumerably infinite number of ways, of which a denumerably infinite number are systematic.

This suggests a question.

QUESTION No. 59: Given a pair of compasses whose opening has been set equal to the distance between a pair of integral points (points having integral coordinates), it is proposed to straddle with this fixed opening of the compasses from one integral point to another, so as to enumerate consecutively, all such points inside and on the boundary of a certain sector of the plane. For what sectors and with what compass settings can this be done?

R.E.G.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Analytic Geometry. By D. R. Curtiss and E. J. Moulton. D. C. Heath and Company, New York, 1930. xiii+338+18 pages. $2.48.

The best way to begin a review of this excellent book is to sketch the plan of the book as the authors state it in the preface. (This preface, by the way, is certainly worth reading because it is so well written and has so many fine ideas concerning text-books). The authors plan to have enough material in the book for a year's course, yet so arranged as to permit selections of topics for shorter courses. They have included many starred sections and also other extra material for superior students. They strive for a clear and simple style and yet for accuracy of statement, for generality in their proofs, for clear-cut definitions. They have an introductory chapter containing formulas and tables from algebra and trigonometry. They use determinants freely, also the reduction and addition formulas for the trigonometric functions. They propose to tie their chapters together, to give more solved illustrative examples than is usual, to arrange the exercises so that the simpler problems precede the difficult.

The answers given at the end of the book are only for the odd-numbered problems. Many pairs of problems have the odd-numbered exercise similar to the even-numbered one. Polar coordinates are introduced at once and developed along with rectangular coordinates. Chapter I reviews elementary curve-plotting such as the student has already met with in algebra and trigonometry; a fuller treatment is then given in Chapter VIII. Chapter XIII, on curve-fitting, gives a practical treatment of the method of least squares. Chapters XIV–XVII contain a brief treatment of solid analytic geometry, including what is used in the calculus.

The authors have succeeded admirably in the plan they outline in their preface. Let us note now some other good points about this text. The book contains a wealth of material and numerous very good figures. It is easy to find any desired topics. No derivatives are used anywhere in the book. The treatments of loci and of symmetry and other parts of curve tracing are excellent. The book contains more advanced topics than many elementary texts, and this in spite of the fact that no space or words are spared in the effort to make everything clear. Many points that trouble the student and must usually be answered by the teacher are taken up in this book either in the text itself, or in the solved problems, or in the footnotes. This makes the book well adapted for home study.

The reader will be struck by some features of the text. For example, there are many ingenious proofs. Thus the derivation of the formulas for rotation of axes is made to depend on the expansion of $\cos(\phi+\theta)$ and $\sin(\phi+\theta)$. The famous nine point circle appears among the examples on page 109. Quite an exhaustive treatment is given of the reduction of conics to standard forms. Brief discussions are given of poles and polars with respect to conics, also of systems of conics, harmonic sets of points, and invariants.

There are some minor criticisms that might be made. For instance, why do the authors debar infinity from the book, and say that a line parallel to the y-axis has no slope? Why do not the authors discuss oblique coordinates? However, the good qualities of the book offset these few criticisms.

The book everywhere shows great care and precision. Thus each equation of a line is spoken of as an equation of the line and not as the equation. The chapter on parametric coordinates is very good and longer than usual. The chapter on curve-fitting will appeal to many teachers as very timely and useful. The treatment of the analytics of space is brief but quite satisfactory. The book has been painstakingly written and shows throughout its pages the earmarks of the long and successful teaching experience of the authors. In every way this is a splendid text, and it ought to appeal to a large clientele as very teachable and well adapted to their needs.

ALAN D. CAMPBELL

A Short History of Mathematics. By Vera Sanford. Houghton Mifflin Company, New York, 1930. xii+402 pages. $3.25

The need for a history of mathematics in one volume has increased during recent years with the growth of interest in this subject and of its importance as a background for the subject matter which is the equipment of secondary school teachers in mathematics. The books in the field twenty-five years ago were adequate for that period but a brief general work of real value will be welcome today as a contribution in supplementing or replacing these earlier works. This one is put out under the editorship of John Wesley Young with an introduction by David Eugene Smith. It will take its place among the standard histories of mathematics for its freshness of treatment, its wide range of subject matter and the sense of authority which it conveys.

The author has had access to and made generous use of the most recent studies and publications along many lines, notably *Sir Isaac Newton, 1727–1927*, edited by F. E. Brasch, *Source Book of Mathematics*, by David Eugene Smith, and *The Rhind Mathematical Papyrus*, by A. B. Chace. Use has been made of many excellent features which have already become familiar through the more extensive and elaborate two-volume work of the distinguished historian and author whose introduction has already been noted. These include maps, chronological details summarized in tables, facsimiles of early works, reproductions of photographs and pictures of instruments. The illustrations are profuse and several are made available for the first time. Such a one is: (p. 96) "Late Use of Roman Numerals in Commercial Accounts in England. From an account book of 1602 (44th year of Elizabeth) now in the collection of Professor David Eugene Smith."

An exceptionally good feature of the work is a summary at the end of each chapter. The one following the chapter on "Men Who Made Mathematics" is a particularly nice biographical characterization and is quoted in full to show the spirit of these summaries as well as the delightful style of the author.

> "What has been given thus far is a picture of the development of mathematics, setting forth the type of people interested in the subject in different countries and at different periods: the priests of ancient Egypt, the merchants and astrologers of Babylonia, the philosophers of Greece, the practical Romans, the scholar-monks of the Dark Ages and of medieval times, the poet-minded writers of India and the Arabs who were as quick to assimilate the knowledge of other nations as they were quick to conquer their territory. Following these were the writers of practical textbooks in the century immediately after the invention of printing and, side by side with them, the speculative scholars who were frequently instructors in the universities and who were interested in the theoretical aspects of the subject. In the seventeenth century there is a group of notable figures, with Descartes, Pascal, Leibniz, and Newton as the most prominent. It is only by contrast with these exceptional figures that the men of the eighteenth century seem less important than their predecessors."

Various devices to aid in seeing the progress of a concept or a symbol are found throughout the book, such as (p. 153) "Methods of Representing Equality," and valuable classifications occur in connection with these. "The Representation of Unknown Quantities" will be cited; it has the sub-headings (p.

156): "By abbreviations and initial letters; By symbols; By exponents with no base given; By exponents with the base given." Such schemes as these help a reader in his first survey and help an instructor in presenting the subject through a text.

While the outstanding facts in connection with each chapter heading are well covered, especial attention may be called to the good general survey under "Commercial Mathematics," a chapter which is original in its treatment throughout and an original contribution from this standpoint, although much of the subject matter is found elsewhere. The space devoted to verbal problems is probably justified by the author in the initial sentence of the chapter: (p. 204) "The problem content of a textbook in arithmetic or in algebra is a measure of the mathematical interests and accomplishments of the group for whom it is written. To a somewhat less degree it is also an indication of the ability and interest of the author, . . . " The author speaks with special authority in this field. Her *History and Significance of Certain Standard Problems in Algebra* is a work which should be consulted generally in connection with the history of mathematics during the centuries which it covers.

The work shows insight and is well-expressed. One instance of this will suffice for many which might be given. (p. 85).

"It is hard to judge the difficulty of computation with number systems other than our own, for we are apt to forget that the actual work is done mentally and that the figures are merely used to record the results. Thus lack of familiarity with the symbols hinders the rapid reading and writing of the numbers that are involved, and accordingly the numerals are branded as being too clumsy for use."

Speculation seems to be irresistible as to what would have happened if something else had not happened, all this from the standpoint of our present knowledge of development along one line. The author states first: (pp. 39–40) "His [Stifel's] *Arithmetica Integra* (1544) contained a comparison of geometric and arithmetic progressions which foreshadowed Napier's invention of logarithms nearly a century later." Later (p. 176) we find: "Stifel (1544) called the upper set of numbers *exponents* and extended both series to the left, thus including negative exponents . . . $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 & 512 \end{bmatrix}$ Close as this was to the idea of logarithms, Napier's invention was the result of a very different line of thought. It is an interesting speculation to try to gauge the influence of the recreational aspects of this subject in inhibiting the use of the idea in the earlier invention of logarithms. Had Napier's invention not come when it did, it is highly probable that logarithms would have come by way of Stifel's exponents." When will authors cease from these useless deductions! The sane attitude to take toward inventions is that found in the following: (p. 111) "The idea of the decimal fraction may indeed be traced to these isolated instances, but it is likely that the man who first developed laws for their use in ordinary computation was not influenced by these early and perhaps unwitting applications." This statement is found in a particularly good account of Stevin's work from the French edition of which the author has made a translation.

While in general no exception can be taken to the contents of the book, erroneous ideas are conveyed at times. (p. 14) " . . . thinker who saw that the two might be combined in a place-value system which would require only ten different characters to write any number, however large" might better read "only as many characters as there are units in the base,—in the common and generally accepted system ten." (p. 257) " . . . fifth century B.C. and from that time to the present the name 'circle-squarers' has been applied to men who have attempted to do things that seem to be impossible." Has the term not rather been applied to men who continue to assert that they can do these things after *science* has proved that they are impossible, making it an opprobrious term rather than a flattering one! (p. 265) "Hippocrates of Chios (c. 460 B.C.) had reduced the problem [duplication of the cube] of solving the equation $x^3=2a^3$ to finding two mean proportionals between the quantities a and $2a$." What Hippocrates did was to reduce the problem to one of plane geometry, namely, inserting two mean proportionals between the edge of the given cube and a line double that edge. The modern form may follow this if it is so stated explicitly. Is there any justification for: (p. 269) "In fact, it [Euclid's Elements] could aptly be called by the name 'General Mathematics'."? The publication of treatises covering different branches of mathematics under one cover does not fit into the blending of these branches, more or less successfully, which is understood by that designation. These instances may all be matters of differences of opinion with regard to an expression to use. The opinion of the reviewer has grown out of long experience in trying to make students in the classroom get the real meaning of just such expressions. Two footnotes relating to the same work should read the same, as they do not in references to a bulletin on pages 154 and 382, the title in the latter case being the correct one, although the middle name of the writer is misspelled.

The contents of the book are: Men Who Made Mathematics; Arithmetic; Commercial Mathematics; Algebra; Verbal Problems; Practical Geometry; Demonstrative Geometry; Trigonometry; Analytic Geometry; Calculus; Theory of Numbers; Calculating Devices; Weights and Measures; The Place of Mathematics in the School Curriculum. General Histories of Mathematics and a Chronological Outline conclude the work. Subtitles of one or two chapters will give a more comprehensive view of the ground covered. Those under "Arithmetic" are: The Name Arithmetic; Number Magic; Number Systems; Computation with Ancient Systems of Numerals; The Abacus; Hindu-Arabic Numerals; Computation with Hindu-Arabic Numerals; Fractions. The Chapter "Demonstrative Geometry" covers so wide a range of topics as: Early Greek Geometry; The Three Famous Problems; Euclid's "Elements"; Non-Euclidean Geometries; The Geometry of Conics; Modern Geometries.

The editor and the publisher are to be congratulated on this addition to the series of books relating to mathematics put out in recent years.

LAO G. SIMONS

Foundations of Potential Theory. (Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, XXXI). By O. D. Kellogg. Julius Springer, Berlin, 1929. ix+384 pages.

What are the desirable characteristics of a book on a classical field of applied mathematics? This question would probably receive somewhat different replies from different groups of scientific workers, such as the experimental physicists, the mathematical physicists, and the mathematicians. However, leaving aside the matter of individual or group preferences, let us consider the question from a broader standpoint. In order that a book of the type referred to may make the maximum contribution to the general development of scientific knowledge, what qualities should it possess? Without any expectation of securing universal agreement, the reviewer of the present volume ventures to formulate his own answer to this question.

First of all the fundamental hypotheses should be closely linked to current physical intuition. That is to say, they should be accurate mathematical formulations of what the general body of working physicists regard as the mechanisms of the physical phenomena involved. Second, the mathematical structure on these foundations should conform to present standards of mathematical rigor, and its generality should at least be adequate for any applications that have been made. If at all feasible, a margin of safety in the matter of generality should be allowed, as past history of the progress of physics shows clearly that the theoretical developments in this field demand an ever increasing degree of generality in the mathematical results that are used.

Professor Kellogg's book, which is one of the important Courant series and the first of American authorship, measures up well to the ideal standard described above. That it should agree with this standard in every particular is hardly to be expected, for even if the author's ideal were identical with the reviewer's, an occasional lapse is to be expected in any human performance. In this connection I am always reminded of one of the bits of dry humor with which Darboux was in the habit of enlivening his lectures. At the end of a rather lengthy and involved reduction he paused and looked quizzically at the final result. Then he closed that part of the discussion with the following peroration, " . . . . ce qui est exact, si je ne me trompe pas. Et pourquoi me tromperai-je? Parce que tout homme est faillible."

I do not mean to imply by the above anecdote that the details which I am about to criticize are necessarily instances of fallibility. They may merely illustrate a difference of opinion between the author and the reviewer. In any event they are what I should classify as minor defects, and I mention them primarily because they are exceptional instances in which the book departs from the ideal standard which I have formulated.

In the first place the reviewer does not approve of the terminology "continuously differentiable," originally introduced on page 97 and used subsequently throughout the book in connection with functions of one or several

variables. Since the word "continuously" modifies the word "differentiable," the implication of the phrase, taken by itself, is that there is something in the *process* of differentiation which is subject to continuous variation. However, the process of differentiation is obviously identical at all points; it is the *result* obtained from that process which exhibits the continuous variation in question. In the case of functions of a single variable, there is no gain in brevity by saying that a function is "continuously differentiable" in place of saying that it has a continuous derivative. In the case of functions of several variables, the statement that the function possesses a continuous directional derivative is somewhat longer but is free from the objection which has been raised.

One may contend that the above criticism has nothing to do with the essential requirements in my ideal standard, since the author accurately defines what he shall mean by functions "continuously differentiable." I should not agree with this contention, however; there is more than a mere matter of linguistic style involved. I prefer to interpret the term "mathematical rigor" in a broad manner. To conform to my idea of it, all definitions in words and all terminology in words should be so phrased as to furnish the maximum aid to clarity of thinking. I do not object to an abbreviated terminology which refers only to some central feature of the mathematical concept described. But I do object to a phrase whose internal structure implies in any way a property foreign to the concept in question.

On somewhat different but essentially similar grounds the reviewer does not approve of the term "piecewise differentiable" introduced on page 97, since the phrase implies that only the existence of a derivative is required, whereas the requirement of the continuity of that derivative is equally important. He objects to such phrases as "reasonably smooth functions" and "sufficiently smooth functions," found on pages 199 and 203 respectively, on the ground of their essential vagueness. In both cases a statement of adequate sufficient conditions would not require much more space. There is far too much shadowy formulation of mathematical requirements in the existing literature of mathematical physics. I always regret to see mathematicians condoning this tendency by even a slight leaning in this direction.

In the case of the discussion on page 203, referred to in the preceding paragraph, where the "sufficiently smooth function" is to be developed in a series of Bessel's functions, there is a further objection to the rather offhand statement as to the requirements to be made. It suggests that the conditions may be found in any standard treatise on the subject, whereas this is not the case. Even in the almost encyclopedic work of Watson, the uniform convergence of the series in question in an interval including the origin is not established, although the physical problem in which this uniform convergence enters goes back to Fourier. The only adequate mathematical discussion up to the present date is to be found in a paper by the present reviewer in the 1911 volume of the Transactions of the American Mathematical Society.

After formulating these criticisms of matters of detail, I do not wish to close

without reiterating my admiration for the general high standard of the book under discussion. It is a splendid piece of work, in which Professor Kellogg's colleagues and friends, as well as the author himself, can justly take pride.

CHARLES N. MOORE

Bernard Bolzano's Schriften. Band 1: Functionenlehre. Published by the Königlichen Böhmischen Gesellschaft der Wissenschaften, Prague, 1930. xx+184+24+vi pages.

In this book there appears in print for the first time the "Functionenlehre" of Bernard Bolzano, as it occurs in a manuscript which has been preserved in the National Library in Vienna.

The occasion for its publication is the nationalistic movement in Czechoslovakia, a movement intended to foster patriotic pride among the inhabitants of this ten-year-old republic which was created as a result of the war. As a part of this movement, efforts have been made to increase popular appreciation of the great artists and scholars which the Czechoslovakian nation has produced. Among these is Bernard Bolzano (1781–1848), famous as priest, philosopher, and mathematician.

Bolzano was born in Prague and spent his entire life in Bohemia. His father was an Italian who emigrated to Prague from northern Italy; his mother was a native of Prague. Appointed professor of the philosophy of religion at Prague at the age of 24, apparently Bolzano's teachings were not acceptable to the ruling powers of the state and consequently he was retired on pension after 15 years. Thus at the age of 40, with a sufficient income for the necessities of life, and forbidden to write on theological topics, he turned to writing on philosophy and mathematics.

Even before his appointment to the chair of theology Bolzano had been interested in mathematics. His first published paper appeared in 1804, a paper in which he attempted to prove Euclid's parallel axiom. He studied the works of Lagrange, Lacroix, and Cauchy, and until his death he continued writing on the foundations of philosophy and mathematics. To date comparatively little has been printed of the mathematical writings which he left in manuscript. The aim of the recently constituted Bolzano-Commission of the Königliche Böhmischen Gesellschaft is to publish the hitherto unprinted works of Bolzano and his scientific correspondence. "Functionenlehre" was chosen as the first of the series to be published by the commission, although it forms only the concluding part of a greater work on "Grössenlehre" which was planned by Bolzano. However, the complete manuscript of this larger work does not exist, and as "Functionenlehre" forms the most interesting part of the work and seems to be ready for publication, it has been published first.

The book begins with a general introduction by Professor K. Petr, describing the life and accomplishments of Bolzano and the purposes of this edition of his works. Then follows Bolzano's text, edited by Professor K. Rychlik with preface and notes. The text consists of three parts: An introduction (pp. 1–12)

on "Relations between two variables;" Chapter 1 (pp. 13–79) on "Continuous and discontinuous functions;" and Chapter 2 (pp. 80–183) on "Derived functions." The introduction is principally concerned with the form of the increment of $y=f(x)$ for certain particular types of functions, and with the limiting values of this increment as Δx approaches zero. Chapter 1 takes up continuity, and in particular left- and right-hand continuity of a function. In Chapter 2, the derivative of a function is defined in the usual way. Left- and right-hand derivatives are studied, but Bolzano decides not to introduce a new notation for them. The mean-value theorem, partial differentiation, integration, and Taylor's series are other topics which are taken up.

Probably the most interesting topic in the entire book is the construction of a function which is continuous in an interval but does not have a derivative at any point of its interval of definition. This example antedates by approximately fifteen years Weierstrass's famous example. The fact that Weierstrass and not Bolzano has received credit for this discovery only serves to emphasize how slight was Bolzano's influence on the development of mathematics. His printed papers on mathematics are few in number. He was not a teacher of mathematics; hence his ideas were not disseminated through the writings and teachings of his pupils. As his discoveries are now brought to light, we see that he was possessed of sound ideas on mathematical subjects, and often of ideas which were not generally known until much later. His criticisms of previously given definitions of continuity are for the most part valid. He points out an error of Galois (p. 96) and evaluates the work of others on Taylor's series. In regard to the function mentioned above, Bolzano proved only that it had no derivative at an everywhere dense set of points. Had he made this result more widely known, someone might have been able to point out the more general property of this function.

The text is printed practically as Bolzano wrote it. The spelling is for the most part as in the original manuscript. The chief difference from modern German seems to be in the use of "ey" for "ei" in words such as *bey*, *beyde*, *Beyspiel*, *frey*, *sey*, *seyn*, *zwey*. A vocabulary on page 1 of the "Notes" gives the modern equivalent of certain phrases used by Bolzano.

The editor has suppressed any desires to edit the text by eliminating the mistakes that Bolzano has made. (A few obvious errors, chiefly numerical errors in the examples, have been corrected.) In the "Notes," which form an exceedingly valuable portion of the book, Professor Rychlik points out which theorems are false and which are true but whose proof is either incorrect or incomplete. Occasionally an error in one theorem invalidates many others. The chief sources of error lie in Bolzano's ignorance of the idea of uniform continuity and his belief that a function of two variables which is continuous in each variable separately is necessarily continuous in both. Incidentally a reading of this text serves to emphasize the superiority of the modern "epsilon-delta" notation. When Bolzano's incorrect proofs are recast in modern notation, the errors in his reasoning are easily seen, whereas the proofs as originally given seem entirely plausible.

The last two pages of the book give a list of misprints. The reviewer noticed a dozen others, all trivial. On pages 22–4 of the "Notes" is a bibliography, which includes the titles of those books from Bolzano's private library which are now in the library of the University of Prague. The title of Young's book as given there does not agree with that given on page 170 of the text. Among the modern books, the third edition of Hobson's *Theory of Functions* might well have been mentioned.

This book will be of much interest to those studying the development of mathematics. It can not, of course, be evaluated as a text on function theory by comparing it with modern texts on that subject. Many things are done now in a better way; but taking into account the times in which he lived, Bolzano seems to have done a good job as a textbook writer. While his rigor is at times not up to modern standards, his manuscript might well serve as the framework of a modern treatise on the theory of functions.

Another interesting point is that Professor Rychlik states (p. 5 of the "Notes") that the proof of the theorem commonly known as the "Bolzano-Weierstrass Theorem" is not to be found either in Bolzano's manuscripts nor in his printed works.

HARRY MERRILL GEHMAN

Statistical Methods for Research Workers. By R. A. Fisher. Third Edition—Revised and Enlarged. Oliver and Boyd, London, 1930. xiii+283 pages+6 tables to be extended outside the text. 15s.

The original edition of this monograph, in a series of "Biological Monographs and Manuals," was greatly appreciated as a product of intimate cooperation with several biological research departments and of daily contact with statistical problems that arise amongst the most advanced of contemporaneous workers.

In the present edition, the section numbers have remained unchanged and the content of most of the sections is the same now as in the first edition. The chief aim of the two revisions seems to be to amplify the principles of statistical estimation, to arouse more interest in the "Method of maximum likelihood," pp. 14, 243 ff., and also to present the most recent developments in statistical method. Two new sections are added to chapter IX which replaced section 6 and example 1, of the first edition. They illustrate further the applicability of the method of maximum likelihood and the quantitative evaluation of information. In his preface, the author laments that, in spite of their practical importance, these new ideas have not met ready acceptance. The summary of principles on pages 270, 271 may show the reason as well as Professor Fisher's confidence in his method, which consists in multiplying the logarithm of the number expected in each class by the number observed, summing for all classes and finding the value of θ for which the sum is a maximum. To read, "In practice one need seldom do more than solve, at least to a good approximation, the

equation of maximum likelihood, and calculate the sampling variance of the estimate so obtained," brings to mind the admonition to beware of any magical process that works though roughly applied.

To consider the "number expected" implies that the type of the universe is known or assumed; to multiply the number observed by the logarithm of the number expected seems like weighting the observations in an unusual way and fitting the products to a selected type of distribution. The choice of the type of universe is far more difficult than curve-fitting, but this is taken as a matter of course, so the special curve-fitting is less impressive. We are told that when the theory of large samples no longer holds, efficient statistics, other than that obtained by the method of maximum likelihood, may fail, but the text has not yet enough in it to convince us of the trustworthiness of small samples without special, independent consideration in each case.

Whether fully warranted or not, some recent writers have attributed to the work of Dr. R. A. Fisher their inspiration for emphasizing the value of small samples to such a degree that we fear that the engineers are in danger of being beset by the small sample specialists somewhat as the psychologists were by the correlaters fifteen years ago.

Amongst the over enthusiastic advocates may be mentioned W. A. Shewhart and F. W. Winters,[1] whose results "so recently obtained" do not differ essentially from those obtained by Thiele,[2] without the questionable emphasis of small samples and without assuming a normal distribution.

As a matter of fact the groundwork and the principles of the socalled "new" work on samples was laid more than forty years ago by Thiele in his monumental work *Almindelig Iagttagelseslaere*,[2] a fact of which the small sample enthusiasts seem to be ignorant. It is therefore not at all surprising to find that Dr. John Wishart,[3] another of the small sample advocates, has published the following statement in a paper in "*The Proceedings of the Royal Society of Edinburgh*":—

"As an estimate of the semi-invariant, κ, Mr. R. A. Fisher has determined a series of symmetric functions k in such a way that the mean value of any k over all samples shall be equal to the corresponding semi-invariant, κ, in the sampled population. The simplest of his results are:

$$
\begin{aligned}
& k_1 = m_1' = N^{-1} \sum_1^N (x), \\
& k_2 = N(N-1)^{-1} m_2, \\
& k_3 = N^2(N-1)^{-1}(N-2)^{-1} m_3, \\
& k_4 = N^2(N-1)^{-1}(N-2)^{-1}(N-3)^{-1}\{(N+1)m_4 - 3(N-1)m_2^2\}.
\end{aligned}
\tag{5a}
$$

[1] Journal of the American Statistical Association, June (1928), p. 144 ff.

[2] Almindelig Iagttagelseslaere (Copenhagen, 1889). Since there is a nice idea wrapped up in Thiele's carefully chosen Danish title, it may be well to translate it, "The general doctrine of observations."

[3] Proceedings of the Royal Society of Edinburgh, Session 1928–29, vol. 49, part 1, No. 7

In the 1889 edition of Thiele's work we find the following formulas:[4]

$$
\begin{aligned}
&\lambda_1 = \mu_1 = m^{-1}\sum_1^m x,\\
&\lambda_2 = m(m-1)^{-1}\mu_2,\\
&\lambda_3 = m^2(m-1)^{-1}(m-2)^{-1}\mu_3,\\
&\lambda_4 = m^3(m-1)^{-1}(m^2-6m+6)^{-1}\{\mu_4 + 6(m-1)^{-1}\mu_2^2\}.
\end{aligned}
\tag{46}
$$

The chief difference in the formulas seems to be that R. A. Fisher has replaced the letters λ, μ, and m in Thiele's notation by the letters k, m, and N. Another quotation from Dr. Wishart[5] may be of interest, viz.,

"Thus the well-known formulas for the moments of the distribution of the mean in samples are summed up, in the notation of this paper, by the formula

$$\kappa(1^r) = \kappa_r/N^{r-1},$$

or in words: The rth semi-invariant of the distribution of the mean is equal to the rth semi-invariant of the sampled population, divided by N^{r-1}."

On page 60 of Thiele's book[2] this result is written as

$$\lambda_r(\mu_1) = m^{1-r}\lambda_r.$$

One, of course, may be dense, but it is difficult to see in what essential way these older formulas differ from those attributed to the reputedly "new" work by the small sample experts. Moreover, Thiele published also in 1889 the error laws (expressed in terms of the presumptive semi-invariants and the number of observations, m) of the semi-invariants, μ, as observed in the sample, and gave besides a numerical table of the sum products of the power sums up to the eighth power so as to facilitate the calculation of the product moments of the semi-invariants.[5] For a concise presentation of the various symmetric functions as employed by the Scandinavian mathematicians in statistical work, interested readers may well look up a review by Mr. Arne Fisher of a book by R. Frisch.[6]

Another phase in which some of the "new" ideas of R. A. Fisher seem to have been anticipated, by several decades at least, by the Scandinavians is in the application he makes of orthogonal functions in fitting regression curves to polynomials. Work of a similar character was also done in 1927 by Birge and Shea in their article in the "University of California Publications in Mathematics"[7] on *A rapid method for calculating the least square solution of a polynomial of any degree.* This reputedly original and new method was, however, described by Gram in his 1879 Danish dissertation on "Raekkeudviklinger."

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[4] Loc. cit., p. 63.

[5] Loc. cit., pp. 61, 62.

[6] Journal of the American Statistical Association, No. 159, Sept. (1927), p. 402.

[7] Vol. 2, No. 5 (issued in March, 1927), pp. 67–118.

Further, Gram gave in the 1915 volume of the "Bulletin de l'Association des Actuaires Suisses," pp. 3–23, a very elegant demonstration of the method together with a numerical table of the generating polynomials up to the fourth order for uneven values of n from 7 to 21 inclusive. The values given in this table, except for an occasional constant multiplier, -1, are exactly the same as the corresponding values of the somewhat more extensive set of tables by Messrs. Birge and Shea, which also cover the even values of n. When they speak of "our new method" we are reminded that "there is nothing new under the sun." The polynomials themselves date back, as mentioned by Gram, even to the eminent Russian mathematician Tchebycheff. The same essential ideas of the Gram polynomials occur also in R. A. Fisher's work, but Dr. Fisher fails to mention either Tchebycheff or Gram.

Finally, we call attention to a paper by J. O. Irwin[8] wherein a comparison is made between Pearson's test for "goodness of fit" and another test devised by Irwin himself. This latter test, which appears superior to the Pearsonian test, is however a special case of Thiele's so-called "Fejlkritik" (error critique) which also is found in the 1889 edition of Thiele's work.

This review has been extended to this length for several reasons:

First, to point out that there is some ground for feeling that Dr. Fisher is too parsimonious in references to other workers outside the fellowship of his own countrymen, and has thus led his followers to remain ignorant of other rich fields.

Second, to call attention to that galaxy of astronomers and actuaries of the first magnitude who have persisted too long in being self-sufficient, publishing only in their native language. We note with appreciation that some of them have started to publish in English.[9]

Third, to serve to arouse some of our mathematicians, especially those who are linguistically inclined, to delve more into the Scandinavian languages.

CHARLES C. GROVE

Curve piane speciali algebriche e trascendenti. Teoria e storia. Volume II.—Curve trascendenti—Curve dedotte da altre con 58 figure illustrative intercalate nel testo. Prima edizione Italiana. By Gino Loria. Ulrico Hoepli, Milano, 1930. xi+439 pp.

The second volume of the first Italian edition of Loria's treatise has followed the first volume with unusual promptness. There are no essential departures from the well-known second German edition of 1910. This volume contains

[8] Journal of the Royal Statistical Society, vol. 92, part 2 (1929).

[9] T. N. Thiele, *Theory of Observations*, 1903 (an elementary extract of the 1889 Danish edition and a translation of a work in Danish issued in 1897). C. L. Charlier, *Researches on the Theory of Probability* (1905). J. F. Steffensen, *Interpolation*, 1927 (the first complete text to appear in English). *Skandinavisk Aktuarietidskrift*, started in 1914, has about two-thirds of its articles in English.

Book VI, consisting of 25 chapters on various transcendental curves, and Book VII with 12 chapters on curves derived from other curves. Some 200 new references, and 60 odd additions to the index of names bring the book thoroughly up to date.

We regret the lack of a subject index for this and for the first volume; and we regret the lack of care in proofreading which makes it highly unsafe to use references without checking them. Nevertheless we are convinced that this is the most useful reference book in this field.

B. H. BROWN

PROBLEMS AND SOLUTIONS

EDITED BY B. F FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3463. *Proposed by D. L. Holl, Ames, Iowa.*

Sum

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\left[\operatorname{sech}\frac{1}{2}\pi(2n-1) + (-1)^{n+1}\tanh\frac{1}{2}\pi(2n-1)\right].$$

3464. *Proposed by I. Maizlish, Centenary College of Louisiana.*

A telegraph pole is in the form of a frustum of a right circular cone of altitude h, diameter of lower base D, and diameter of upper base d. A rope, of radius r, is wound spirally around this pole from the bottom to the top, covering the whole pole. A device is constructed which unwinds the rope—beginning at the top. If the angular velocity, w, with which the rope is being unwound is a function of the length of the rope already unwound, find the time it will take the device to unwind the whole rope.

3465. *Proposed by J. P. Dalton, University of Witwatersrand, Johannesburg, South Africa.*

Prove that for all positive integral values of n

$$\sum_{i=n}^{2n}(-1)^{2n-i}\frac{(2i+2)!}{i!(2n-i)!(2i-2n)!} = 2^{n-1}(n+1)(2n+1)(2n^2+7n+4).$$

3466. *Proposed by Leonard M. Blumenthal, Rice Institute.*

The lineo-linear expression, which we may exhibit in the matrix form

	x_1	x_3	x_5	x_{2n-1}
$\bar{x}_2$	$n-1$	-1	$-1 \cdots$	-1
$\bar{x}_4$	-1	$n-1$	$-1 \cdots$	-1
$\bar{x}_6$	-1	-1	$n-1 \cdots$	-1
.	. . . . .			
.	. . . . .			
.	. . . . .			
$\bar{x}_{2n}$	-1	-1	-1	$n-1$

where the coefficient of $x_{2k-1}\bar{x}_{2j}$ is found at the intersection of the kth column and the jth row, and $\bar{x}_{2j}$ is the conjugate of x_{2j}, is invariant under translations applied to either set x_{2i-1}, x_{2i} of the variables, and also under the rotations $y_i = tx_i$, $t = e^{i\theta}$ $(i = 1, 2, \cdots 2n)$.

Considering the complex numbers x_{2i-1}, x_{2i} $(i = 1, 2, \cdots, n)$ as defining the vertices of two ordered plane n-gons, find a geometric interpretation of this invariant.

3467. *Proposed by J. Rosenbaum, Melford, Conn.*

Solve in positive integers: $2x^2+2x+1=y^2$.

3468. *Proposed by Charles A. Rupp, Pennsylvania State College.*

Show that the determinant of n^2 elements in the upper left corner of the Pascal triangle

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & \cdot & \cdot \\ 1 & 2 & 3 & \cdot & \cdot & \\ 1 & 3 & \cdot & \cdot & & \\ 1 & \cdot & \cdot & & & \\ \cdot & \cdot & & & & \\ \cdot & & & & & \end{array}$$

has the value unity.

SOLUTIONS

2856. [1920, 377]. *Proposed by O. S. Adams, U. S. Coast and Geodetic Survey.*

Show that for the real domain defined by $+1 > x > -1$, s a positive integer,

$$\frac{1}{(1-x^s)^{1/s}} \int_0^x \frac{dx}{(1-x^s)^{(s-1)/s}} = x + \sum_{n=1}^{n=\infty} \frac{2(s+2)(2s+2)\cdots(ns-s+2)}{(s+1)(2s+1)\cdots(ns+1)} x^{ns+1}$$

and

$$\frac{1}{(1-x^s)^{(s-1)/s}}\int_0^x \frac{dx}{(1-x^s)^{1/s}} = x + \sum_{n=1}^{n=\infty} \frac{n!s^n}{(s+1)(2s+1)\cdots(ns+1)}x^{ns+1}.$$

Solution by Morgan Ward, California Institute of Technology.
Let

$$\int_0^x \frac{dx}{(1-x^s)^{(s-1)/s}} = x(1-x^s)^{1/s}F(u), \tag{1}$$

$$\int_0^x \frac{dx}{(1-x^s)^{1/s}} = x(1-x^s)^{(s-1)/s}G(u), \quad u = x^s.$$

The solution of the problem then reduces to showing that

$$F(u) = 1 + \sum_{n=1}^{\infty} \frac{2(s+2)(2s+2)\cdots(ns-s+2)}{(s+1)(2s+1)\cdots(ns+1)}u^n, \tag{2}$$

$$G(u) = 1 + \sum_{n=1}^{\infty} \frac{n!s^n}{(s+1)(2s+1)\cdots(ns+1)}u^n. \tag{3}$$

Assume that $+1>x>-1$. Then we may differentiate both sides of (1) with respect to x, obtaining

$$\frac{1}{(1-x^s)^{(s-1)/s}} = [(1-x^s)^{1/s} - x^s(1-x^s)^{(1-s)/s}]F + sx^s(1-x^s)^{1/s}\frac{dF}{du},$$

or

$$su(1-u)(dF/du) + (1-2u)F = 1. \tag{4}$$

If equation (4) has a solution expressible as a power series,

$$F(u) = \sum_{n=0}^{\infty} a_n u^n,$$

it is evident that $a_0=1$. By substituting this expression in equation (4) and equating coefficients of u^n, we obtain the recurrence formula

$$a_n = a_{n-1}(ns-s+2)/(ns+1) \qquad (n = 1, 2, \cdots),$$

from which (2) follows immediately. Then, since (2) is absolutely convergent for $|u|<1$, it is a solution of (4).

In a similar manner, we find that $G(u)$ satisfies the differential equation

$$su(1-u)(dG/du) + (1-su)G = 1. \tag{5}$$

As before, we find that the only solution expressible as a power series in u is

$$G(u) = \sum_{n=0}^{\infty} b_n u^n,$$

where

$$b_0 = 1, \quad b_n = b_{n-1}ns/(ns+1) \quad (n = 1, 2, \cdots ;$$

from this (3) follows at once. Again, since (3) is absolutely convergent for $|u|<1$, it is a solution of (5). It is evidently unnecessary to restrict s to be an integer, but it is necessary that s be positive in order that $|x|<1$ may imply that $|u|<1$.

3124. [1925, 138]. *Proposed by M. B. Porter, University of Texas.*

$f(x)$ and $\phi(x)$ are polynomials and all the zeros of $\phi(x)$ are real. Let $P(x)=\phi(x)[f^{(i)}(x)]^r$ where $f^{(i)}$ stands for the ith derivative of $f(x)$.

Prove that $[P'(0)]^2<P''(0)P(0)$ is a sufficient condition that $f(x)$ has imaginary zeros. Newton's test[1], $C_i^2 \leqq C_{i-1}C_{i+1}$, where C_i is the coefficient of x^i in $f(x)$, is a special case.

I. Solution by Otto Dunkel, Washington University.

All the polynomials considered here are required to have real coefficients. The theorems of the problem may be deduced in various ways from well known and easily proved theorems. One of these theorems is as follows:

If the derivative $P'(x)$ of the polynomial $P(x)$ has $2i$ imaginary roots, then $P(x)$ has at least $2i$ imaginary roots.

We proceed to apply this lemma. Replace x in $P(x)$ by $h+y^{-1}$, where h is a real quantity. The equation $P(x)=0$ is replaced by

$$P(h)y^p + P'(h)y^{p-1} + \tfrac{1}{2}P''(h)y^{p-2} + \cdots = 0, \tag{1}$$

where p is the degree of $P(x)$, $p \geqq 2$. The $(p-2)$th derived equation is

$$p(p-1)P(h)y^2 + 2(p-1)P'(h)y + P''(h) = 0. \tag{2}$$

If now

$$[P'(h)]^2 < p(p-1)^{-1}P(h)P''(h), \tag{3}$$

the equation (2) has imaginary roots; and, by the repeated use of the lemma, we see that (1) has at least a pair of imaginary roots. It then follows that $P(x)=0$ has at least a pair of imaginary roots.

If then h is real and

$$[P'(h)]^2 < P(h)P''(h), \quad \text{or} \quad = P(h)P''(h) \neq 0, \tag{4}$$

it is obvious that (3) must be true. For the same value of h the test (3) is better than (4), since it does not require as much as (4). If then (3) or (4) is true $P(x)=0$ has at least one pair of imaginary roots. If $P(x)\equiv\phi(x)[f^{(i)}(x)]^r$, $r>0$, where $\phi(x)$ has no imaginary roots, then $f^{(i)}(x)$ must have some imaginary roots, and it then follows that $f(x)$ must have at least a pair of imaginary roots.

[1] Cf. Netto: *Vorlesungen über Algebra*, vol. I, p. 234, edition of 1896.

If we take $P(x)=f^{(i-1)}(x)$, where

$$f(x) = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n, \quad C_0 C_n \neq 0, \tag{5}$$

and set $h=0$, then (3) becomes

$$C_i^2 < \frac{n-i+1}{n-i}\,\frac{i+1}{i} C_{i-1}C_{i+1}. \tag{6}$$

If (6) is satisfied for any one of the $n-1$ values of i, then the equation (5) has at least one pair of imaginary roots. Also (6) is obviously satisfied if

$$C_i^2 < C_{i-1}C_{i+1}, \text{ or if } C_i^2 = C_{i-1}C_{i+1} \neq 0. \tag{7}$$

If two consecutive coefficients of (5) are zero, it is well known that there must be at least one pair of imaginary roots. This fact may also be proved by the above reasoning. A similar remark applies to (1).

Another method for obtaining (4) directly from (1) is as follows: If $r_1, r_2, \cdots, r_p$ are the roots of (1), $-\sum r_i = P'(h)/P(h)$, $\sum r_ir_j = P''(h)/2P(h)$, and

$$[\sum r_i]^2 - 2\sum r_ir_j = \sum r_i^2. \tag{8}$$

The rest of the proof follows easily.

A third method of obtaining (4), without the use of (1), is to employ the identity

$$\frac{P'(h)}{P(h)} = \sum_{i=1}^{i=p} \frac{1}{h-r_i}. \tag{9}$$

By taking the derivatives of both sides and making a slight transformation in the result, we find

$$[P'(h)]^2 - P(h)P''(h) = \sum{}'[(h-r_1)(h-r_2)\cdots(h-r_p)]^2, \tag{10}$$

where the accent mark means that in each of the p products a linear factor is omitted.

A fourth form of proof may be obtained by multiplying $P(x)$ by a linear factor $x-a$, where a is real, and then choosing for a a value which makes one of the intermediate coefficients of the product zero. Then apply Descartes' rule of signs in the usual way to detect imaginary roots. See the article by Oliver D. Kellogg, *A necessary condition that all the roots of an algebraic equation be real*, Annals of Mathematics, ser. 2, vol. 9, 1908, p. 97.

The tests (6) and (7) may be called quadratic tests. By a slight variation of the first method a variety of other tests may be obtained.

II. Solution by Paul Wernicke, Washington, D. C.

Sylvester's extension of Newton's rule, which is capable of some simplification *if from the start* we assume Netto's[1] Case I, leads to $a_n^2{}_{-1}-2a_{n-2}a_n>0$ if $f(z)$ is to have real roots only. This is $[f'(0)]^2-f''(0)f(0)>0$. With r a positive integer, writing $[f^{(i)}(z)]^r=F$, we have

$$(P')^2 - P''P = [(\phi')^2 - \phi''\phi]F^2 + [(F')^2 - F''F]\phi^2$$

the first term of which cannot be negative for $z=0$ by the condition on ϕ. Therefore the bracket of the second term is negative and $F(z)=0$ or $[f^{(i)}z]^r=0$; hence $f^{(i)}(z)=0$ has complex roots. By the last paragraph on p. 208 (Netto, loc. cit., III) this is a sufficient condition for all preceding derivatives and $f(z)$ itself to have complex roots.

Since it is always possible to choose three coefficients of ϕ so that (for $r=1$) $(P')^2-P''P$ for any i becomes equal to one of the expressions (15) referred to, Newton's "test" is a special case of the above.

3412. [1930, 157]. *Proposed by P. R. Rider, Washington University.*

The vertex of a right circular cone of semi-vertical angle α is the center of a unit cube, the axis of the cone coinciding with a diagonal of the cube. Find the area of that part of the conical surface which is inside the cube.

Partial Solution by V. F. Ivanoff, San Francisco, Cal.

Let the center O of the cube be the origin of rectangular coordinates; let the positive z-axis lie upon the sem-diagonal OC, where C is a vertex of the cube; and let the plane zOx be perpendicular to a face through C. The equation of this face is

$$z = -2^{1/2}x + 3^{1/2}2^{-1}; \tag{1}$$

and the equations of the cone are

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = \rho\cot\alpha. \tag{2}$$

From (1) and (2) we have

$$\rho = \frac{3^{1/2}}{2(\cot\alpha + 2^{1/2}\cos\theta)}. \tag{3}$$

The differential of the conical surface inside the cube may be written

$$dP = \frac{1}{2}\cdot\rho d\theta\cdot\rho\csc\alpha = \frac{3d\theta}{8\sin\alpha(\cot\alpha + 2^{1/2}\cos\theta)^2}. \tag{4}$$

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[1] Loc. cit.; p. 231, bottom; p. 234, bottom; p. 234, expressions (15).

Hence

$$P = \frac{9}{4 \sin \alpha} \int_0^{\pi/3} \frac{d\theta}{(\cot \alpha + 2^{1/2} \cos \theta)^2}, \tag{5}$$

where P is the surface of one sheet of the cone within the cube. This integral may be transformed into a rational integral by known methods of the calculus.

The result (5) is true only when $\tan \alpha \leqq 2^{3/2}$. If $\tan \alpha > 2^{3/2}$ the conical surface of one sheet intersects all faces of the cube and the problem becomes more complicated.

3413. [1930, 157]. *Proposed by P. R. Rider, Washington University.*

The axis of a right circular cylinder of radius r coincides with a diagonal of a unit cube. Find the area of that part of the cylindrical surface which is inside the cube.

Partial Solution by V. F. Ivanoff, San Francisco, Cal.

With the same disposition of the axes as in the solution of 3412, we have again the equation

$$z = -2^{1/2}x + 3^{1/2}2^{-1}, \tag{1}$$

and the equations of the cylinder,

$$x = r \cos \theta, \quad y = r \sin \theta. \tag{2}$$

Hence for the intersection of the cylinder with the face (1) we have

$$z = -2^{1/2}r \cos \theta + 3^{1/2}2^{-1}. \tag{3}$$

The differential of the cylindrical surface between the xOy-plane and the face is

$$dP = rd\theta \cdot z = r(3^{1/2}2^{-1} - 2^{1/2}r \cos \theta)d\theta; \tag{4}$$

and the total surface within the cube is

$$\begin{aligned} P &= 6r \int_0^{\pi/3} (3^{1/2} - 2^{3/2}r \cos \theta)d\theta \\ &= 2 \cdot 3^{1/2}r(\pi - 3 \cdot 2^{1/2}r). \end{aligned} \tag{5}$$

The result (5) is true only when all the elements of the cylinder lie within the cube, i.e. when $r \leqq 2^{-\frac{1}{2}}$. In the cases where $2^{-1/2} < r < 2^{\frac{1}{2}} \cdot 3^{-\frac{1}{2}}$ the problem is more complicated. If $r \geqq 2^{\frac{1}{2}} \cdot 3^{-\frac{1}{2}}$ the cube is inside the cylinder.

Also solved by A. Pelletier.

Note by the Editors: The proposer of these two problems (3412, 3413) states that they are special cases of problems which arise in certain statistical studies. See the article by Paul R. Rider, *On the distribution of the ratio of mean to standard deviation in small samples from non-normal universes*, Biometrika, vol. XXI, Dec., 1929, pp. 139–141, and the references cited therein. See also a similar

problem in an article by G. A. Baker, *Random sampling from non-homogeneous populations*, Metron, vol. 8, no. 3, Feb., 1930, pp. 74–76.

3414. [1930, 157]. *Proposed by B. C. Wong, Berkeley, California.*

Prove or disprove:

$$\sum_{i=0}^{t}\left[\frac{(r-1)!(r-2i)}{(r-i)!i!}\right]^2=\frac{(2r-2)!}{r!(r-1)!},$$

where $t=(r-2)/2$ if r is even and $t=(r-1)/2$ if r is odd.

Solution by Emma T. Lehmer, Brown University.

Using binomial coefficient notation we are to prove that

$$f(r)\equiv\sum_{i=0}^{[(r-1)/2]}\left[\binom{r}{i}\frac{r-2i}{r}\right]^2=\frac{1}{r}\binom{2r-2}{r-1}. \tag{1}$$

Changing i to $r-i$ in (1) and adding the result to (1), we get

$$f(r)=\frac{1}{2}\sum_{i=0}^{r}\left[\binom{r}{i}\frac{r-2i}{r}\right]^2$$

or

$$f(r)=\frac{1}{2}\sum_{i=0}^{r}\binom{r}{i}^2-2\sum_{i=0}^{r-1}\binom{r}{i+1}\binom{r-1}{i}+2\sum_{i=0}^{r-1}\binom{r-1}{i}^2. \tag{2}$$

To evaluate these sums, consider the identity

$$(1+x)^r(1+x)^q\equiv(1+x)^{r+q}.$$

Equating coefficients of x^{r-s} on both sides we obtain

$$\sum_{i=0}^{q}\binom{r}{s+i}\binom{q}{i}=\binom{r+q}{q+s}. \tag{3}$$

As special cases of (3) we have

$$\sum_{i=0}^{r}\binom{r}{i}^2=\binom{2r}{r}\quad\text{and}\quad\sum_{i=0}^{r-1}\binom{r}{i+1}\binom{r-1}{i}=\binom{2r-1}{r}.$$

Substituting these results in (2) we obtain

$$\begin{aligned}f(r)&=\frac{1}{2}\binom{2r}{r}-2\binom{2r-1}{r}+2\binom{2r-2}{r-1}\\&=\binom{2r-2}{r-1}\left[\frac{2r-1}{r}-2\frac{2r-1}{r}+2\right]=\frac{1}{r}\binom{2r-2}{r-1}.\end{aligned}$$

Also solved by W. I. Miller.

3415. [1930, 157]. *Proposed by L. S. Johnston, University of Detroit.*

Let A, C, O, C', A' be collinear points, with $OA=OA'$, $OC=OC'$; let l_1, l_2, and l_3 be lines perpendicular to AA' through A, O, and A' respectively. With any point P on l_2 as center and PC as radius a circle is drawn. Let d_1 and d_2 be the diameters determined by the intersections of this circle with l_1 and l_3. Find the envelope of d_1 and d_2.

I. Solution by A. Pelletier, Montreal, Canada.

Taking AA' for the x-axis, l_2 for y-axis, and letting $(0, m)$ be the coördinates of any point, P, on l_2, $OA=OA'=a$, $OC=OC'=c$, we have for the equation of the circle having center P and passing through C, $x^2+(y-m)^2=m^2+c^2$, or $x^2+y^2-2my-c^2=0$.

Solving this equation simultaneously with the equations of the lines $x=\pm a$ for points of intersection and then writing the equations of the lines passing through the two pairs of points symmetric with respect to the center, $(0, m)$, we have the equations of the two diameters d_1 and d_2:

$$y - m = \pm (m^2 + c^2 - a^2)^{1/2}a^{-1}x.$$

Eliminating m, the variable parameter, between these two equations and their first derivative equation with respect to m, we obtain for the equation of the envelopes of d_1 and d_2,

$$a^{-2}x^2 - (c^2 - a^2)^{-1}y^2 = 1,$$

which is the equation of an hyperbola when $c>a$, and an ellipse when $c<a$.

II. Solution by Otto Dunkel, Washington University.

Set $OA=a$, $C'O=OC=c<a$, and consider one of the variable lines d_1 cutting l_2 and l_1 in P and Q, respectively, where $PQ=PC$. Let the foot of the perpendicular from C to d_1 be N, and produce CN to D so that $CN=ND$. Draw the line $C'D$ cutting d_1 in T, and then draw CT. It will be shown that d_1 is tangent at T to the ellipse having foci C', C, and vertices A', A. For the triangles CPQ and AON are similar and isosceles, as easily follows from the fact that C, O, P, N are concyclic, and similary for Q, A, C, N. Hence $ON=OA=a$. Moreover, $C'T+TC=2ON=2a$, and TC' and TC make equal angles with d_1. From symmetry the same reasoning applies to d_2.

If $c>a$, the same kind of reasoning shows that the envelope is an hyperbola. Also solved by Rufus Crane, P. S. Dwyer, J. H. Neelley, and the Proposer.

3416 [1930, 157]. *Proposed by William Hoover, Columbus, Ohio.*

If a circumscribed triangle to a given circle have two sides fixed and the third variable, to determine the envelope of its circumcircle.

Solution by William I. Miller, University of Pittsburgh.

Using oblique Cartesian coordinates, place the triangle with the fixed vertex at the origin, and the fixed sides along the positive ends of the axes. Let the

fixed angle of the triangle be 2ϕ. Let the center of the circle be the point (h, h), and its radius $r=h \sin 2\phi$. Let 2α be the angle exterior to the triangle which the variable side makes with the X-axis, and let the X and Y intercepts of this variable line be a and b, respectively. By trigonometry,

$$a = h + h \cos 2\phi + r \tan \alpha = 2h \cos \phi \cos (\alpha - \phi) \sec \alpha,$$

$$b = h + h \cos 2\phi + r \cot (\alpha - \phi) = 2h \cos \phi \sin \alpha \csc (\alpha - \phi).$$

The equation of the circle through $(0, 0)$ $(a, 0)$, and $(0, b)$ is

$$f(x, y) \equiv x^2 + y^2 + 2xy \cos 2\phi - ax - by = 0.$$

To obtain the equation of the envelope, substitute the above values for a and b and eliminate α between $f=0$ and $\partial f/\partial \alpha = 0$. The latter equation reduces to

$$\cos \alpha \csc (\alpha - \phi) = (x/y)^{1/2}.$$

From this,

$$a = 2h[1 + (y/x)^{1/2} \sin \phi], \quad b = 2h[1 + (x/y)^{1/2} \sin \phi].$$

Only the positive values of the radicals are used, since $0<\phi<\alpha<90°$ and $a>2h$, $b>2h$. Substituting these values for a and b in the equation of the circle, we obtain an equation which is factorable:

$$(x + y + 2(xy)^{1/2} \sin \phi)(x + y - 2(xy)^{1/2} \sin \phi - 2h) = 0.$$

The first factor, if rationalized, represents two conjugate imaginary lines, whose real point of intersection is the origin. The second factor, if rationalized, represents a circle tangent to the coordinate axes with center at $(h \sec^2\phi, h \sec^2\phi)$ and radius $2h \tan \phi$. But as it stands, this factor represents only the larger of the two arcs determined by the points of contact with the axes; the smaller arc is obtained if the given circle be escribed to rather than circumscribed about the given triangle.

To construct the envelope circle geometrically, let $OABC$ be the rhombus circumscribing the given circle, with OA and OC the fixed lines. At A erect a perpendicular to OA, intersecting the diagonal OB at D; then D is the center and AD the radius of the required circle. The larger of the two arcs AC, together with the point O, is the required envelope.

Also solved by Rufus Crane, A. Pelletier, O. J. Ramler, Mabel M. Young, and the Proposer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Associate Professor Karl Menger, of the University of Vienna, who is lecturing at Harvard University during the present semester, will lecture at Rice Institute during the second semester this year. At Rice Institute he will give an undergraduate course on "Metrical Geometry" and a graduate course on "Theory of Dimension and Curve Theory."

Professor O. D. Kellogg, of Harvard University, has been appointed exchange professor to a group of three middle-western colleges for the second half of the present academic year. He will spend about five weeks lecturing at each. The three colleges are: Carleton College, Northfield, Minnesota; Knox College, Galesburg, Illinois; and Colorado College, Colorado Springs, Colorado.

Princeton University has conferred an honorary doctorate on President K. T. Compton of the Massachusetts Institute of Technology.

The following persons have been appointed National Research Fellows in Mathematics for the academic year 1930–31 (This list includes reappointments, and may not be complete): R. P. Agnew, A. C. Berry, Leonard Carlitz, L. W. Cohen, C. C. Craig, H. T. Engstrom, D. H. Lehmer, S. B. Littauer, N. H. McCoy, E. J. McShane, Gordon Pall, W. T. Reid, N. W. Rutt, W. J. Trjitzinsky, Jacob Yerushalmy.

Mr. O. J. Farrell has been appointed Rockefeller International Research Fellow in Mathematics.

John V. Atanasoff, of the University of Wisconsin, has been appointed assistant professor of mathematics at the Iowa State College.

Professor Ella E. Bernstorf, of Friends University, Wichita, Kansas, has been appointed dean of women at the State Teachers College, California, Pa.

Dr. Nat Edmonson, Jr., of Rice Institute, has been appointed assistant professor of mathematics at the Texas Technological College.

Assistant Professor J. D. Eshleman, of the University of Pennsylvania, has been appointed acting professor of mathematics at Heidelberg College, Tiffin, Ohio.

Mr. Ivan L. Hebel, of the Colorado School of Mines, has been promoted to an assistant professorship of mathematics.

Mr. E. R. Heineman has been appointed assistant professor of mathematics at the Texas Technological College.

Dr. A. W. Hobbs, professor of applied mathematics at the University of North Carolina, has been appointed dean of the College of Liberal Arts of that University.

Professor Alfred Hume, of the University of Mississippi, has been appointed professor of mathematics at Southwestern College, Memphis, Tenn.

Dr. P. W. Ketchum, of the University of Illinois, has been promoted to be an associate in mathematics.

Dr. F. W. Kokomoor, of the University of Florida, has been promoted to an associate professorship of mathematics.

Assistant Professor H. I. Lane, of the University of South Dakota, has been promoted to an associate professorship of mathematics.

Assistant Professor J. D. Leith, of the University of North Dakota, has been promoted to an associate professorship of mathematics.

Assistant Professor Elizabeth LeStourgeon, of the University of Kentucky, has been promoted to an associate professorship of mathematics.

Professor Gertrude I. McCain, of Westminster College, has been appointed professor of mathematics at State College, Radford, Va.

Dr. Wilhelm Maier, on leave from the University of Frankfort, has been appointed visiting professor of mathematics at Purdue University for the year 1930–31.

Dr. Morris Marden has been appointed assistant professor of mathematics in the Extension Division of the University of Wisconsin, at Milwaukee.

Dr. C. G. Phipps has been promoted to an associate professorship of mathematics at the University of Florida.

Dr. P. G. Robinson has been promoted to an associate professorship of mathematics at Iowa State College.

Assistant Professor H. M. Sheffer has been promoted to an associate professorship of philosophy at Harvard University.

Professor J. A. G. Shirk, of Kansas State Teachers College, who has been granted a year's leave of absence, is studying mathematical statistics at Stanford University. Professor W. H. Hill is acting head of the department during his absence.

Dr. R. G. Smith, of the University of Kansas, has been elected associate professor of mathematics at Kansas State Teachers College.

Associate Professor F. W. Sparks has been promoted to a professorship of mathematics at the Texas Technological College.

Dr. C. W. Strom has been promoted to a professorship of mathematics at Luther College.

Assistant Professor D. J. Struik has been promoted to an associate professorship of mathematics at the Massachusetts Institute of Technology.

One Hundred Per Cent Membership

The membership in the Association is now about 2200, including 137 institutional members. These figures indicate one hundred percent increase over the total charter membership in 1916. Presumably this also indicates that among the mathematics faculties in many institutions the Association membership has doubled during these fourteen years. However, in some institutions, probably in very many, *all* members of the mathematics staff and the institution itself have belonged to the Association from the outset. Such a one hundred percent membership has been maintained, for instance, by the University of Chicago. It is urgently desired by the membership committee to ascertain all of the institutions of which this is now true, and to this end the committee on membership requests the cooperation of each mathematical staff. Will the secretary or some representative of each such department transmit to the Secretary of the Association at Oberlin, Ohio, the following information:

(1) Has your department at present a one hundred percent individual membership in the Association?

(2) Has your institution an institutional membership in the Association?

(3) Will you cordially invite any non-members in your group to join the Association?

(4) Will you present to your institution the desirability of becoming an institutional member of the Association?

Membership in the Association is a mark of professional standing and a contribution to the promotion of mathematical interests in America. Members are entitled to receive all publications of the Association at cost, including the Carus Monographs and the Rhind Mathematical Papyrus. The Association now has eighteen sections distributed over the country so that any member may attend a meeting within reasonable distance; and all meetings, sectional and national, are fully reported in the American Mathematical Monthly.

The near approach of the Annual Meeting in Cleveland, Ohio, is an auspicious time to secure new members for 1931. The Secretary will supply application blanks on request, and they should be returned to him not later than December 27. Why not adopt the slogan—"One Hundred Percent Membership."

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CORRIGENDA

Volume XXXVI, 1929:

P. 545, first line, for "K" read "k."
Pp. 532, 21st line, delete "s" in "New Yorks."
P. 558, under "Swezy," for "E.D." read "G.D."

Volume XXXVII, 1930:

P. 39, 4th line frim the bottom, for "S. Pelletier" read "A. Pelletier."
P. 162, 28th line, omit "e" in "Whiteman."
P. 203, 19th line, for "W.M." read "W. I."
P. 250, 11th line from bottom, for "cos" read "sin."
P. 386, 4th line from bottom, for "G" read "J."
P. 404, last line, for "μ^*" read "μ_*."
P. 486, 8th line, for "substraction" read "subtraction."
P. 504, 11th line, for "octatredron" read "octahedron."
P. 510, 7th line from bottom, for "Ramley" read "Ramler."

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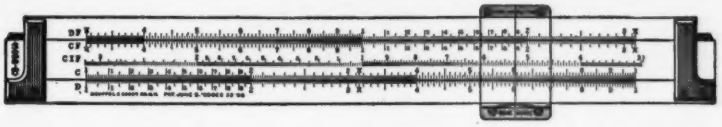

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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.

INDIANA, Earlham College, May 2-3.

IOWA, Ames, Iowa, May 2-3.

KANSAS, February 15.

KENTUCKY, Lexington, Ky., April 5.

LOUISIANA-MISSISSIPPI, Cleveland, Miss., March 7-8.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, May 10.

MICHIGAN, Ann Arbor, Mich., March 22.

MINNESOTA, Carleton College, May 17.

MISSOURI, Columbia, Mo., November 28.

NEBRASKA, Peru, Neb., May 9.

OHIO, Columbus, Ohio, April 3.

PHILADELPHIA, Philadelphia, Pa., Nov. 29.

ROCKY MOUNTAIN, Denver, Colo., April 11-12.

SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, Calif., March 8.

TEXAS.

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Mathematics

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